Contagion and fire sales
in banking networks[*]

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Abstract

The paper develops a theoretical framework to analyze the connection between the structure of a banking network and its resilience to systemic shocks. We base our analysis on the model of interbank contagion proposed by Cifuentes, Ferrucci and Shin (2005), that accounts for the impact of illiquid assets' fire sales. We elaborate on this model mainly along three directions: (i) analytically proving, in a more general setting, the existence of an equilibrium and the convergence of the algorithm that can be used to compute it; (ii) extending the scope of the simulations (e.g., including an assessment of the resilience of different stylized network topologies and a sensitivity analysis); (iii) generalizing the model to deal with the case where more than one illiquid asset is available on the market.

JEL   D85,   C62,   G21,   G28

Keywords   financial networks, contagion, liquidity, fire sales, systemic risk

Contents

1 Introduction

The structure of lending relationships among banks determines the degree of interconnectedness of the financial system and is relevant for the stability of individual institutions and the system as a whole. The latest financial crisis brought to the attention of both academics and policymakers the importance of taking into account interconnections among financial institutions when assessing systemic risk. Regulatory initiatives have been undertaken accordingly. For instance, the Financial Stability Board has included measures of interconnectedness in its framework on systemically important financial institutions.

From a theoretical point of view, network analysis provides a convenient way to model linkages among financial institutions. Since the seminal paper by Allen and Gale (2000), a network perspective has been increasingly used in Economics and Finance and network analytical tools have become part of the policymaker's toolkit, especially for financial stability purposes (see Gai, 2013).

Recall that a network is simply a collection of nodes, connected with one another by arcs, or edges (see Newman, 2010, and Jackson, 2008, for a complete introduction to network analysis and its applications in Economics). When modeling a banking system with networks, nodes represent banks, while arcs stand for their bilateral exposures on the interbank market. The way in which these links are arranged (the network topology) may significantly influence how the system responds to exogenous shocks. Indeed, in this simplified setting, network topology entirely accounts for the system's "emerging properties", i.e. those features of the system that cannot be inferred from properties of its individual components (individual banks), but only from the way in which these units are related to (interact with) each other.

The academic literature that studies banks with network analytical tools has grown at a fast pace over the last few years. For the sake of simplicity, a rough taxonomy may be devised, discerning two main streams: (i) papers that either study network formation with game theoretical instruments or model contagion in the banking system through different channels, often working on simulated data; (ii) papers that study statistical properties and network metrics of real banking systems, either with a descriptive focus or, again, for contagion analysis purposes.

The present paper relates to the former stream of literature. In particular, it develops a framework that can be used to study how the network topology of a banking system may influence its resilience to systemic shocks. To this end, we refine and generalize the model of interbank contagion proposed by Cifuentes, Ferrucci and Shin (2005) – CFS from now on – a pioneering paper on the interaction between two important channels of contagion:[1] direct balance-sheet exposures

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[1]In this paper we adopt a rather broad and informal definition of contagion as the transmission of (severe) negative shocks affecting one or more banks to other banks in the system. This is in line with "what has evolved into the most common usage of the term contagion – the transmission of an extreme negative shock in one country to another country (or group of countries)" (Forbes, 2012).

between banks and price deterioration of bank assets due to fire sales. We elaborate on this model mainly along three directions: (i) we provide the single asset model with complete analytical foundations, presenting a constructive proof of the existence of an equilibrium, along with the convergence of the algorithm used to find it (Section 3; in this connection, we relax the authors' assumption on the demand function); (ii) we extend the scope of the simulations in the single asset case, in order to answer the research question concerning the resilience of different stylized network topologies to exogenous systemic shocks (Section 4; we also provide a sensitivity analysis of the model to its main parameters, along with related policy implications); (iii) we generalize the model to deal with the case where more than one illiquid asset is available on the market, a more realistic assumption that may significantly increase both theoretical and computational complexity (Section 5). On the other hand, in line with the vast majority of the related literature, we do not model possible strategical anticipation of the default of a bank by other institutions (banks can only take action once they are hit by a shock or by contagion) and we do not model all relevant channels of contagion (in particular, we do not consider liquidity hoarding). These extensions are left for future research.

Consistent with the literature, the main results of our paper highlight the crucial role of the network topology in determining how the banking system may be affected by systemic shocks. In particular, simulation results confirm that networks with a star or core-periphery structure may be more resilient,[2] but are more fragile with respect to targeted shocks that affect core banks (Section 4.2). Concerning the distribution of the initial shock, we also show that, even in the case of homogeneous banks (i.e., banks with identical balance-sheets), the impact of contagion following an exogenous shock of fixed aggregate magnitude depends on the number of banks among which the shock is distributed. The dependence is no-linear and varies according to the network topology (Section 4.3).

Furthermore, we show that the network structure also affects the relative importance of the two main parameters of the model (i.e., the degree of liquidity of non-interbank assets and the requirement on the leverage ratio) in the transmission of the shock. For instance, in the star topology the two parameters are equally important, while in the circle and totally interconnected configurations either the former or the latter plays a preeminent role (Section 4.5). As a consequence, the relative effectiveness of policy instruments (e.g. capital or liquidity requirements) employed to tackle different risks in the banking system may depend on the topology of the interbank network.

Finally, when several illiquid assets are available, we model the liquidation strategy of each bank as a solution to a convenient optimization problem. In this way we are able to: (i) analytically justify the behavioral rule assumed in the CFS model for the single-asset case; (ii) highlight possible discrepancies between microfounded optimal strategies and liquidation strategies that are often assumed as sensible behavioral rules. In particular, since bank assets are marked to market,

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[2]See Section 4.1 for a description of the stylized network configurations that we consider in our study.

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the liquidity of an asset is not an intrinsic quality of the asset (fully described by the elasticity of its price) from the point of view of a bank that has to deleverage, but depends on both the composition of the bank's portfolio and the liquidation strategies adopted by the other banks in the system.

The remainder of the paper is organized as follows. Section 2 reviews the relevant literature, with a particular focus on the presentation of the CFS model. Section 3 presents our formalization of the model in the case of a single illiquid asset, with the proof of the existence of an equilibrium and the convergence of the algorithm. Section 4 discusses the results of simulation exercises in the single asset case based on the theoretical framework built in the previous section, while Section 5 generalizes that framework to the case of multiple illiquid assets. Finally, Section 6 concludes.

2   Literature review

In recent years there has been a rapid development of the literature on the modelization of contagion in financial networks. A comprehensive review of this stream of literature can be found in Amini and Minca (2012); here we mention only the papers which are essential for the development of our framework.

Eisenberg and Noe (2001) deal with default of financial institutions in a single clearing mechanism; they prove the existence of a clearing payment vector in a network in which the obligations are determined simultaneously and give mild regulatory conditions under which the clearing vector is unique. Moreover, they develop an algorithm that computes the payment vector and highlights the flow of contagion among the nodes of the network.

Cifuentes, Ferrucci and Shin (CFS, 2005) develop a stylized model in which a contagion mechanism overlaps a fire sale effect. In this context there are two channels of contagion: on one hand, banks can be affected by direct exposure on the interbank market, where some of their counterparties can default; on the other hand, banks hold similar portofolios, whose market value can be depressed by a fire sale mechanism. We will describe in detail the model developed by CFS in Section 2.1, since the present paper builds on that framework.

Gauthier, Lehar, Souissi (2010) deal with macroprudential capital allocation and systemic risk. They compare different models, among which a network model based on CFS, calibrated on the data relative to the Canadian banking network. In particular, they investigate the relationship between macroprudential capital allocation and systemic risk.

The importance of the structure of the lending relationships among banks for the stability of the financial system has been studied by many authors (see, e.g., Allen and Gale, 2000, and Allen, Babus and Carletti, 2010). Some recent papers add to this stream of literature studying, from a theoretical perspective, the resilience of banking networks characterized by different stylized configurations (e.g., autarchy, star, circle or full diversification; see Cabrales, Gottardo and Vega-Redondo, 2013).

2.1 CFS model

The starting point of this work is the paper by Cifuentes, Ferrucci and Shin (2005), who deal with liquidity risk in a network of financial institutions and explore the effects of asset fire sales. They consider a system of interconnected institutions (which can be thought of as stylized banks) subject to a regulatory constraint on their leverage ratio, calculated marking-to-market the value of assets. The institutions in the network can hold interbank claims and two kinds of assets (one liquid and one illliquid);[3] moreover, they can borrow funds from the interbank market and collect deposits. When the system is hit by an exogenous shock, some of the institutions may go in distress and become unable to meet the capital requirement; in this case they can ease their balance sheet positions by selling part of their non-interbank assets. If the market demand for such assets is not perfectly inelastic, then sales by distressed institutions can lower the corresponding price. Since the regulatory ratio is computed using marked-to-market assets, a further round of endogenously generated sales can occur, possibly leading to contagious failures. CFS lays the basis for the description of contagious failures driven by asset fire sales in a network of financial insitution holding similar portfolios of assets.

The model considers a system of N interconnected banks. Each institution can hold interbank assets and liabilites, decribed by a $N \times N$ matrix

$$L = (L_{ij})_{1 \leq i,j \leq N},$$

where the entry L_{ij} is the face value of the liability of bank i to bank j. By definition $L_{ij} \geq 0$ and $L_{ii} = 0$ for all i, j. Let L_i be the face value of the total interbank liabilities of bank i, i.e.

$$L_i = \sum_j L_{ij},$$

and π_{ij} be the proportion of bank i's liabilities associated with bank j:[4]

$$\pi_{ij} = \begin{cases} L_{ij}/L_i & \text{if } L_i > 0 \\ 1/N & \text{if } L_i = 0. \end{cases}$$

We denote by x_i the market value of bank i's interbank liabilities, which can be lower than L_i because the bank can be unable to repay all of its debts. The model assumes that all the interbank claims have equal seniority, so the payments made by a bank in distress are proportional to its notional liabilities: the borrower i pays the amount $x_i \pi_{ij}$ to the lender j.

In addition, bank i is endowed with a certain amount c of liquid assets and a fixed holding e of illiquid assets, which can be sold at the market price p (which

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[3]In the model, the degree of liquidity is represented by a parameter accounting for price elasticity, as shown in equation (4).

[4]The definition of π_{ij} when $L_i = 0$ ensures that row-sums of π equal one in any case.

is endogenously determined by the model). Moreover, bank i collects d_i units of deposits. Then the equity value of bank i is given by

$$pe_i + c_i + \sum_j x_j \pi_{ji} - x_i - d_i. \tag{1}$$

Besides equal priority of interbank claims, the model assumes two additional conditions: limited liability of the shareholders, which implies that bank i's equity value is nonnegative; and priority of debt over equity, which implies that the equity value is strictly positive only if $x_i = L_i$.

For a fixed value of p, based on the previous assumptions the vector of interbank payments is given by

$$x_i = \min \left\{ L_i, pe_i + c_i + \sum_j x_j \pi_{ji} - d_i \right\}; \tag{2}$$

either a bank is able to pay its liabilities in full (L_i), or it pays the marked-to-market asset value.[5] Equation (2) shows that bank i's equity value depends on the payments x_j of the other institutions, as well as on the market price p of the illiquid assets. The price p is endogenously determined by the model and depends on the amount of assets liquidated in the market by the banks in distress.

In this model each institution is subject to a capital adequacy contraint: the ratio between the equity value, where the interbank assets and liabilities are calculated in terms of the expected payments, and the marked-to-market value of the assets held by the bank must be above a pre-specified threshold $\bar{r}$. If a bank violates the capital requirement, then it starts selling its assets to ease its balance-sheet position (in this kind of models this is the only feasible option, since raising fresh capital is not allowed). Let t_i be the units of the liquid asset sold by the distressed institution (at full price 1) and s_i be the units of illiquid asset sold by bank i (at price $p \leq 1$). Then, after the sale of t_i units of liquid assets and s_i units of illiquid assets, the regulatory leverage ratio reads

$$r_i = \frac{pe_i + c_i + \sum_j x_j \pi_{ji} - x_i - d_i}{p(e_i - s_i) + (c_i - t_i) + \sum_j x_j \pi_{ji}} \geq \bar{r}. \tag{3}$$

In equation (3), the denominator is the marked-to-market value of the assets after the sale of t_i and s_i: it is assumed that assets are sold for cash (giving $t_i + ps_i$) and that cash does not attract capital requirement, so the ratio takes into account only the residual assets held by the bank.

The goal of the model is to determine an equilibrium solution (x, s, p), which is defined as follows.

Definition 2.1 *The triple (x, s, p), with $x =$ vector of payments, $s =$ vector of sales of illiquid assets, $p =$ price of the illiquid asset, is an equilibrium if the following conditions are satisfied:*

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[5]Actually, when deposits are added to the balance-sheet, the quantity $pe_i + c_i + \sum_j x_j \pi_{ji} - d_i$ may take negative values if the counterparties of bank i do not repay their liabilities in full. Then, (2) should be replaced by $x_i = \max \left\{ 0, \min \left\{ L_i, pe_i + c_i + \sum_j x_j \pi_{ji} - d_i \right\} \right\}$, as in (10).

1. *for all i, $x_i = \min \left\{ L_i, pe_i + c_i + \sum_j x_j \pi_{ji} - d_i \right\}$;*

2. *for all i, s_i is the smallest amount of illiquid assets sold by bank i which guarantees that the regulatory constraint is satisfied, upon selling all the liquid asset. If $r_i < \bar{r}$ for all values of s_i, then s_i is set equal to e_i.*

3. *the equilibrium price satisfies $p = \xi^{-1}(\sum_i s_i)$, where ξ^{-1} denotes the inverse demand function.*

In CFS the inverse demand curve for the illiquid asset is assumed to be exponential,

$$p = \exp\left(- \alpha \sum_i s_i \right), \tag{4}$$

where α is a fixed exogenous parameter mirroring the semi-elasticity of the price. An equilibrium price is defined as a value p such that the aggregate sale $s(p) = \sum_i s_i(p)$ of illiquid assets equals the demand $\xi(p)$, $s(p) = \xi(p)$.

The existence of an equilibrium triple (x, s, p) is a nontrivial issue, even in a stylized interbank network. In fact, the problem is highly nonlinear in the three unknowns, which are defined by implicit conditions. The paper by CFS deals with two subproblems separately:

- on one hand, it states that equation (2) can be solved, for fixed p and s, using a fixed point theorem, as in Eisenberg and Noe (2001);

- on the other hand, it explores the existence of the equilibrium price p (and simultaneously of the optimal quantity s), keeping the payment vector x fixed. The paper provides some conditions under which, for fixed x, the problem admits an equilibrium price lower than 1 (i.e. the trivial solution in which no fire sale occurs) and gives a graphical intuition of the price dynamics.

However, the work by CFS does not address explicitly the existence of the whole equilibium solution (x, s, p). In addition, when non-bank deposits are added to the balance-sheet, the arguments used in Eisenberg and Noe (2001) do not apply any longer and the proofs need to be deeply modified. The paper by CFS states that the equilibrium can be determined computationally through an iterative procedure, whose results are shown in a simulation exercise. In the following section we will present a complete analytical foundation of CFS model, including a constructive proof of the existence of the equilibrium, which will be the basis for our simulations.

3 Analytical foundations for the single asset case

In this section we provide a complete analytical foundation for a CFS-like model, presenting a constructive proof of the existence of an equilibrium, along with the convergence of the algorithm used to find it. None of the following proofs is

shown or sketched in CFS. As we will show in this section, the existence of an equilibrium is not trivial at all, despite the highly stylized model.

The main result is stated in Theorem 3.8: we construct a non-increasing sequence of prices p^k and a non-increasing sequence of payment vectors x^k which converge to an equilibrium of the problem. Moreover, we describe in detail the algorithm we will implement in the simulations, which generalizes CFS to the case of a general demand function and explicit exogenous shocks. The iterative procedure is composed of two intertwining algorithms: a modification of Eisenberg and Noe's algorithm to compute the interbank payment vector and a fixed point argument to determine the price of the illliquid asset through a fire sale mechanism.

3.1 Notation and preliminaries

1. Sales of illiquid assets. Let $\xi = \xi(p)$ be a demand function for the illiquid asset, with $p \in (0, 1]$. We assume that ξ is continuous, invertible and such that ξ^{-1} is non-increasing. Let $s = s^x(p)$ be the supply function, defined as the sum of the units of illiquid asset sold by each bank in the system, $s^x(p) = \sum_i s_i^x(p)$, where $s_i^x(p)$ is defined as follows:

$$s_i^x(p) = \begin{cases} e_i & \text{if } x_i < L_i \\ \max\big\{\min\{e_i, g_i^x(p)\}, 0\big\} & \text{otherwise,} \end{cases} \tag{5}$$

and

$$g_i^x(p) = -\frac{(1 - \bar{r})e_i}{\bar{r}} + \frac{x_i + d_i - (1 - \bar{r})\sum_j x_j \pi_{ji} - c_i}{\bar{r}p}. \tag{6}$$

Note that $g_i^x(p)$ is defined as the amount of illiquid assets a bank in distress needs to liquidate to match exactly the capital ratio, after selling all the available liquid assets ($t_i = c_i$), when the realized payments in the system are described by x. If the regulatory condition holds for bank i with no need to sell illiquid assets, then $s_i = 0$ by construction. If the value of the ratio remains below the threshold $\bar{r}$ for any sale of illiquid assets, then s_i is set equal to e_i. By construction, $s_i^x(p) \in [0, e_i]$. We keep the superscript x in s to remind that the value of $s_i^x(p)$ depends on x: the amount of assets a bank needs to liquidate to fulfill the regulatory requirement is a function of the realized payments in the interbank network, possibly lower than the face values of the claims. Note that for all i and x, $s_i^x(p)$ is a non-increasing continuous function of p and that there exist two values $0 \leq q_i^1 \leq q_i^2 \leq 1$ such that $s_i^x(p) = e_i$ for $p \leq q_i^1$, $s_i^x(p) = 0$ for $p \geq q_i^2$ and $s_i^x(p) = g_i^x(p)$ for $p \in (q_i^1, q_i^2)$.

Although the price p can vary in $(0, 1]$ *a priori*, actually we can restrict the range of feasible prices to a smaller interval, which depends on the network structure and on the initial shock which hits the system.

- The upper bound p_0 of the price set depends on the extent of the initial shock which hits the system, possibly triggering a fire sale mechanism. In this setting an exogenous shock is modeled by cancelling part of the illiquid

assets held by the banks of the network; it can be concentrated on one financial institution or widespread over a number of banks.

The after-shock price p_0 is a function of the magnitude of the shock. If it were equal to 1 regardless of the extent of the shock, we might obtain some incongruous results: a large shock on the illiquid assets of one bank could lead to a small overall effect, since the after-shock quantity of illiquid asset in the bank's portfolio could not be sizable enough to trigger a fire sales mechanism; on the other hand, a smaller shock to the same bank could generate more contagious effects, since in this case the bank's after-shock endowment of illiquid asset could suffice to trigger widespread fire sales, if liquidated.

Let σ be the overall amount of illiquid assets cancelled by the exogenous shock. We then assume that, when the shock hits the system, the price of the illiquid asset drops to the after-shock price

$$p_0 = \xi^{-1}(\sigma) = \xi_\sigma^{-1}(0), \tag{7}$$

where we introduce the additional notation

$$\xi_\sigma^{-1}(s) := \xi^{-1}(\sigma + s).$$

Since the magnitude σ of the initial shock is a fixed parameter of the model, for notational simplicity we will often write ξ^{-1} instead of ξ_σ^{-1}. Note that the parameter σ affects the function ξ_σ^{-1} only by horizontal shifts (it determines the starting point of the inverse demand function, corresponding to $s = 0$); the shape of the function is the same for all values of the initial shock.

Note that p_0 is the price of the illiquid assets prior to the spread of the contagion and the asset fire sales. Since the fire sale mechanism can only depress the market value of the assets, any feasible price p must be lower than or equal to p_0.

- On the other hand, the lowest feasible price corresponds to the case in which all the banks in the system liquidate their endowment of illiquid assets in full:

$$p_{\min} = \xi^{-1}\left(\sum_i e_i\right). \tag{8}$$

We define

$$\Psi_\sigma^x(p) = \xi_\sigma^{-1}(s^x(p)), \tag{9}$$

with $s^x(p)$ as in (5). The map Ψ_σ^x will be often denoted by Ψ^x for simplicity. From the previous observations we deduce that

$$\Psi_\sigma^x : [p_{\min}, p_0] \to [p_{\min}, p_0].$$

Since it is the composition of two non-increasing functions, Ψ^x is non-decreasing in p. By Tarski's fixed point theorem (see Theorem A.1) we can deduce the existence and uniqueness of the maximal fixed point of Ψ_σ^x, for a given x:

Lemma 3.1 *For fixed x, there exists a unique $p^* \in [p_{\min}, p_0]$ such that*

$$\Psi^x_\sigma(p^*) = p^*$$

and $p^* \geq q$ for all $q \in [p_{\min}, p_0]$ satisfying $\Psi^x_\sigma(y) = y$.

2. Clearing payments. Having fixed a price p for the illiquid asset, the vector x is a clearing payment vector if it is a fixed point of the map

$$\Phi_p : [0, L_1] \times \ldots \times [0, L_N] \to [0, L_1] \times \ldots \times [0, L_N]$$

defined as

$$\Phi_p(x) = \max\{\min\{L, \Pi'x + pe + c - d\}, 0\}. \tag{10}$$

The existence of a maximal fixed point of Φ_p can be obtained by Tarski's fixed point theorem (see Theorem A.1).

Lemma 3.2 *For fixed p, there exists a unique vector $x \in [0, L_1] \times \ldots \times [0, L_N]$ such that*

$$x = \Phi_p(x)$$

and that $x \geq y$ for all $y \in [0, L_1] \times \ldots \times [0, L_N]$ satisfying $y = \Phi_p(y)$.

Note that the function Φ_p can be written in an equivalent form, which will be convenient to describe the algorithm. Let

$$D_p(x) = \{i \in 1, \ldots, N : (\Phi_p(x))_i < L_i\}$$

and let

$$(\Lambda_p(x))_{ij} = \begin{cases} 1 \text{ if } i = j \text{ and } i \in D_p(x) \\ 0 \text{ otherwise.} \end{cases}$$

For fixed p, given $x, y \in [0, L_1] \times \ldots \times [0, L_N]$ we define

$$
\begin{aligned}
F_{p,y}(x) &= \Lambda_p(y)\big(\min\{L_i, \max\{0, \Pi'(\Lambda_p(y)x + (I - \Lambda_p(y))L) + pe + c - d\}\}\big) \\
&\quad + (I - \Lambda_p(y))L.
\end{aligned} \tag{11}
$$

By construction we have

$$F_{p,x}(x) = \Phi_p(x), \text{ for all } x. \tag{12}$$

Note that, for fixed p, y, the map $F_{p,y}(\cdot)$ is non-decreasing.
We introduce a definition which will be useful in the proofs.

Definition 3.3 *The vector x is a supersolution for Φ_p if*

$$x \geq \Phi_p(x).$$

We now prove a lemma on the monotonicity of $F_{p,y}$ which will be useful in what follows.

Lemma 3.4 *The function $F_{p,y}(x)$ is non-decreasing:*

(a) *in p, for fixed x, y;*

(b) *in y, for fixed p, x;*

Proof.

(a) Notice first that, if $p \leq q$, $D_p(y) \supseteq D_q(y)$, as $\Phi_p(y)$ is non-decreasing in p for fixed y. As a consequence, $\Lambda_p(y) \geq \Lambda_q(y)$ component-wise. We can distinguish three cases:

- if $(\Lambda_p(y))_{ii} = (\Lambda_q(y))_{ii} = 0$, then $(F_{p,y}(x))_i = L_i = (F_{q,y}(x))_i$;
- if $(\Lambda_p(y))_{ii} = (\Lambda_q(y))_{ii} = 1$,

$$
\begin{aligned}
(F_{p,y}(x))_i &= \min\left\{L_i, \max\left\{0, \sum_j \pi_{ji}((\Lambda_p(y))_{jj}x_j + (1 - (\Lambda_p(y))_{jj})L_j) \right.\right. \\
&+ \left.\left. pe_i + c_i - d_i\right\}\right\} \\
&\leq \min\left\{L_i, \max\left\{0, \sum_j \pi_{ji}((\Lambda_q(y))_{jj}x_j + (1 - (\Lambda_q(y))_{jj})L_j) \right.\right. \\
&+ \left.\left. qe_i + c_i - d_i\right\}\right\} = (F_{q,y}(x))_i,
\end{aligned}
$$

as $(\Lambda_p(y))_{jj}x_j + (1 - (\Lambda_p(y))_{jj})L_j \leq (\Lambda_q(y))_{jj}x_j + (1 - (\Lambda_q(y))_{jj})L_j$ and $pe_i \leq qe_i$;

- if $(\Lambda_p(y))_{ii} = 1$ and $(\Lambda_q(y))_{ii} = 0$,

$$
\begin{aligned}
(F_{p,y}(x))_i &= \min\left\{L_i, \max\left\{0, \sum_j \pi_{ji}((\Lambda_p(y))_{jj}x_j + (1 - (\Lambda_p(y))_{jj})L_j) \right.\right. \\
&+ \left.\left. pe_i + c_i - d_i\right\}\right\} \leq L_i = (F_{q,y}(x))_i.
\end{aligned}
$$

(b) If $y \leq z$, then $D_p(y) \supseteq D_p(z)$, since $\Phi_p(y)$ is non-decreasing in y for fixed p. Therefore, $\Lambda_p(y) \geq \Lambda_p(z)$ component-wise. The proof is then on the same line of (a). ∎

We finally present a definition of *equilibrium* coherent with our framework.

Definition 3.5 *A pair $(x^*, p^*) \in [0, L_1] \times \ldots [0, L_N] \times [p_{\min}, p_0]$ is said to be an equilibrium if*

$$
x^* = \Phi_{p^*}(x^*) \quad and \quad p^* = \Psi^{x^*}(p^*).
$$

3.2 The algorithm

We assume that an interbank system as in the previous sections is hit by an exogenous shock σ on the illiquid assets. Let e_i^0 be the initial endowment of illiquid assets for bank i and e_i be the after-shock one. Then $\sigma = \sum_i e_i^0 - \sum_i e_i$. The algorithm runs as follows:

- Step 0 (Initialization): we set

$$x^0 = L, \ p^0 = p_0, \ s^0 = (0, \ldots, 0), \ t^0 = (0, \ldots, 0).$$

- Step k, $k \geq 1$: in the previous step we determined the payment vector x^{k-1}, the amount of illiquid assets sold by each bank $s^{k-1} = (s_1^{k-1}, \ldots, s_N^{k-1})$ and the corresponding price p^{k-1}. Moreover we calculated the amount of liquid assets sold by the banks in the network $t^{k-1} = (t_1^{k-1}, \ldots, t_N^{k-1})$.

  Now, we first compute the value of the leverage ratio for each bank:

$$r_i^k = \frac{p^{k-1}e_i + c_i + \sum_j x_j^{k-1}\pi_{ji} - x_i^{k-1} - d_i}{p^{k-1}(e_i - s_i^{k-1}) + (c_i - t_i^{k-1}) + \sum_j x_j^{k-1}\pi_{ji}}.$$

  We define the vector of indicator functions I^k as $I_i^k = \chi_{r_i^k < \bar{r}}$, i.e. $I_i^k = 0$ if bank i complies with the capital requirement, $I_i^k = 1$ otherwise.

  ◇ If $t_i^{k-1} = c_i$ and $s_i^{k-1} = e_i$ for all i such that $I_i^k = 1$, then we terminate the algorithm: either a bank is satisfying its regulatory ratio (i.e. $I_i^k = 0$), or it has sold all the available assets (i.e. $I_i^k = 1$, $t_i^{k-1} = c_i$ and $s_i^{k-1} = e_i$). In this case there is no further sale of illiquid assets and the corresponding price remains constant. We set:

$$x^* = x^{k-1} = L, \ p^* = p^{k-1}, \ t^* = t^{k-1}, \ s^* = s^{k-1}. \tag{13}$$

  ◇ If, for some i, $I_i^k = 1$ and $s_i^{k-1} < e_i$, then we compute the amount of assets sold by the banks in distress, the resulting price of the illiquid assets and the payment vector.

    ○ **Computation of p^k, s^k and t^k.** Namely, we first calculate the units of liquid assets sold by each bank, which is given by

$$t_i^k = \max\left\{\min\left\{\frac{d_i + x_i^{k-1} - (1 - \bar{r})(p^{k-1}e_i + c_i + \pi'x^{k-1})}{\bar{r}}, c_i\right\}, 0\right\}.$$

    The formula above states that bank i sells the smallest amount of available liquid assets which guarantees the fulfilment of the capital adequacy condition. If the regulatory ratio is still violated after selling the entire endowment c_i, then bank i needs to start selling its illiquid assets.

    Now, s^k and p^k are defined as follows:

$$s^k = \sum_i s_i^k,$$

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$$s_i^k = s_i^{x^{k-1}}(p^k) = \begin{cases} e_i & \text{if } x_i^{k-1} < L_i \\ \max\left\{ \min\{e_i, g_i^k(p^k)\}, 0 \right\} & \text{otherwise,} \end{cases}$$
(14)

with

$$g_i^k(p) = -\frac{(1-\bar{r})e_i}{\bar{r}} + \frac{x_i^{k-1} - (1-\bar{r})\sum_j x_j^{k-1}\pi_{ji} - c_i}{\bar{r}p},$$
(15)

and p^k is the greatest fixed point of

$$\Psi^k(p) = \xi^{-1}(s^k(p)) = \xi^{-1}(s^{x^{k-1}}(p)) \text{ in } [p_{\min}, p_0].$$

In order to find p^k numerically, we implement a sub-algorithm starting from the initial guess $P_0 = p_0$. Then for $n \geq 1$ we set the recursive condition $P_n = \Psi^k(P_{n-1})$. By Lemma A.2, the sequence $\{P_n\}$ converges to p^k, so we can stop the iterations when $|\Psi^k(P_n) - P_n|$ is lower than a pre-assigned tolerance level ε.

○ **Computation of** x^k. We define x^k as the greatest fixed point of the map $F_{p^k, x^{k-1}}$, i.e.

$$F_{p^k, x^{k-1}}(x^k) = x^k$$

and $x^k \geq z$ for all z such that $F_{p^k, x^{k-1}}(z) = z$. In order to find a numerical approximation of x^k, we start a sub-algorithm from $X_0 = L$ and we set $X_n = F_{p^k, x^{k-1}}(X_{n-1})$ for $n \geq 1$. By Lemma A.2, the sequence $\{X_n\}$ converges to x^k, so we can stop the iterations when $|F_{p^k, x^{k-1}}(X_n) - X_n|$ is below a pre-assigned tolerance level ε.

Lemma 3.6 *If $p^{k+1} \leq p^k$ and $x^k \geq \Phi_{p^k}(x^k)$, then*

$$x^{k+1} \leq x^k \quad \text{and} \quad x^{k+1} \geq \Phi_{p^{k+1}}(x^{k+1}).$$

Proof. Let $\hat{x}^{k+1}$ and x^{k+1} be the greatest fixed points of the maps $F_{p^k, x^k}(\cdot)$ and $F_{p^{k+1}, x^k}(\cdot)$, respectively, i.e.

$$F_{p^k, x^k}(\hat{x}^{k+1}) = \hat{x}^{k+1}, \ \hat{x}^{k+1} \geq z \text{ for all } z \text{ such that } z = F_{p^k, x^k}(z),$$

and

$$F_{p^{k+1}, x^k}(x^{k+1}) = x^{k+1}, \ x^{k+1} \geq z \text{ for all } z \text{ such that } z = F_{p^{k+1}, x^k}(z).$$

Note that $\hat{x}^{k+1}$ and x^{k+1} are well-defined by Theorem A.1. We firstly show that

$$x^{k+1} \leq \hat{x}^{k+1} \leq x^k,$$
(16)

thus getting the first part of the proof. Recall that $F_{p,y}(\cdot)$ is non-decreasing and, by Lemma 3.4, $F_{p^{k+1}, x^k}(y) \leq F_{p^k, x^k}(y)$ for all y. Then, by Lemma A.3,

$$F_{p^{k+1}, x^k}^{(n)}(L) \leq F_{p^k, x^k}^{(n)}(L).$$

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By passing to the limit as $n \to +\infty$ on both sides of the previous inequality and using Lemma A.2, we obtain $x^{k+1} \leq \hat{x}^{k+1}$.

The second inequality can be proved analogously. By Lemma 3.4, $F_{p^k,x^k}(y) \leq F_{p^k,x^{k-1}}(y)$ for all y. As a consequence,

$$F_{p^k,x^k}^{(n)}(L) \leq F_{p^k,x^{k-1}}^{(n)}(L),$$

from which, letting n tend to infinity, $\hat{x}^{k+1} \leq x^k$; this concludes the proof of (16).

It remains to show that x^{k+1} is a supersolution, i.e. $x^{k+1} \geq \Phi_{p^{k+1}}(x^{k+1})$. We have just shown that $x^{k+1} \leq x^k$; by Lemma 3.4 we get

$$F_{p^{k+1},x^k}(x^{k+1}) \geq F_{p^{k+1},x^{k+1}}(x^{k+1}).$$

Since $F_{p^{k+1},x^k}(x^{k+1}) = x^{k+1}$ and $F_{p^{k+1},x^{k+1}}(x^{k+1}) = \Phi_{p^{k+1}}(x^{k+1})$, the proof is complete. ∎

Lemma 3.7 *If $x^k \leq x^{k-1}$, then $p^{k+1} \leq p^k$.*

Proof. We recall that, for fixed p and i, $s_i^k(p)$ and $g_i^k(p)$ are defined as in (14) and (15). If $x_i^k = L_i$, the assumption $x^k \leq x^{k-1}$ implies that $x_i^{k-1} = L_i$. There follows that $g_i^{k+1}(p) \geq g_i^k(p)$, hence $s^{k+1}(p) \geq s^k(p)$. On the other hand, if $x_i^k < L_i$ then $s_i^{k+1}(p) = e_i$ by definition, so $s_i^{k+1}(p) \geq s_i^k(p)$ for all p. Since ξ^{-1} is non-increasing, we get that

$$\xi^{-1}(s^{k+1}(p)) \leq \xi^{-1}(s^k(p)), \quad \text{i.e. } \Psi^{k+1}(p) \leq \Psi^k(p).$$

Since Ψ^{k+1} is non-decreasing, by applying Lemma A.3 we get

$$(\Psi^{k+1})^{(n)}(p_0) \leq (\Psi^k)^{(n)}(p_0) \quad \text{for all } n \geq 1.$$

By Lemma A.2, we can pass to the limit as n tends to infinity and get

$$p^{k+1} \leq p^k,$$

as desired. ∎

Theorem 3.8 *There exist a non-increasing sequence of prices $\{p_k\}$ converging to a price p^* and a non-increasing sequence of payment vectors $\{x_k\}$ converging to a vector x^*, such that (p^*, x^*) satisfy*

$$x^* = \Phi_{p^*}(x^*) \quad \text{and} \quad p^* = \Psi^{x^*}(p^*) = \xi_\sigma^{-1}(s^{x^*}(p^*)). \tag{17}$$

The pair (x^*, p^*) is then an equilibrium according to Definition 3.5.

Proof. Let $p_0 = \xi^{-1}(\sigma)$ and $x^0 = L$. For $k \geq 1$, we define p^k as the greatest fixed point of the function $\Psi^k = \Psi^{x^{k-1}}$ and then x^k as the greatest fixed point of the function $F_{p^k,x^{k-1}}$.

We can show by induction that

$$p^{k+1} \leq p^k, \quad x^{k+1} \leq x^k, \quad \text{and} \quad x^{k+1} \geq \Phi_{p^{k+1}}(x^{k+1}). \tag{18}$$

- For $k = 0$ we have $x^0 = L$, $p^0 = p_0$ and $x^0 = L \geq \Phi_{p_0}(x^0)$. Then by definition $p^1 \leq p^0 = p_0$ and $x^1 \leq x^0 = L$. Moreover, $x^1 \geq \Phi_{p^1}(x^1)$ by Lemma 3.6.

- Now, we assume that $p^k \leq p^{k-1}$, $x^k \leq x^{k-1}$ and $x^k \geq \Phi_{p^k}(x^k)$ and we show that this implies (18).
  By Lemma 3.7, we have that $x^k \leq x^{k-1}$ implies $p^{k+1} \leq p^k$. Now, by Lemma 3.6, we deduce that $x^{k+1} \leq x^k$ and $x^{k+1} \geq \Phi_{p^{k+1}}(x^{k+1})$, as desired.

The sequence $\{x^k\}$ is non-increasing and takes values in the set $[0, L_1] \times \ldots [0, L_N]$, hence it admits a limit $x^* \in [0, L_1] \times \ldots [0, L_N]$. The sequence $\{p^k\}$ is non-increasing and $p^k \in [p_{\min}, p_0]$ for all k; hence it admits a limit $p^* \in [p_{\min}, p_0]$. It remains to show that (x^*, p^*) satisfies (17).

By the definition of x^{x+1}, the monotonicity of $F_{p^{k+1}, x^k}(\cdot)$ and the fact that x^k is a supersolution, we get

$$x^{k+1} = F_{p^{k+1}, x^k}(x^{k+1}) \leq F_{p^{k+1}, x^k}(x^k) \leq F_{p^k, x^k}(x^k) = \Phi_{p^k}(x^k) \leq x^k.$$

Since $x^k \to x^*$ and $x^{k+1} \to x^*$, by comparison $\{\Phi_{p^k}(x^k)\}$ admits a limit and

$$\lim_k \Phi_{p^k}(x^k) = x^*.$$

The function $\Phi_p(x)$ is continuous in $(x, p) \in \mathcal{D} = [0, L_1] \times \ldots \times [0, L_N] \times [p_{\min}, p_0]$ by construction, hence

$$x^* = \lim_k \Phi_{p^k}(x^k) = \Phi_{p^*}(x^*).$$

By definition of p^k we have

$$\lim_k \Psi^k(p^k) = \lim_k p^k = p^*. \tag{19}$$

It remains to show that $p^* = \Psi^{x^*}(p^*)$. Recall that $\Psi^x(p) = \xi^{-1} \circ s^x$. In this case we cannot claim that Ψ is continuous in (x, p) in the whole domain $\mathcal{D}$, because s is not continuous as a function of x (see equation (5)). However, we can show that the continuity of Ψ on a subset of $\mathcal{D}$ is sufficient to prove the desired result.

Let

$$\mathcal{A} = \left\{ \left(\prod_{i=1}^N A_i \right) \times [p_{\min}, p_0] : \ A_i = \{L_i\} \vee A_i = [0, L_i) \text{ for all } i \right\}.$$

We can show that:

1. $\Psi = \Psi^x(p)$ is continuous in B for all $B \in \mathcal{A}$;

2. there exist $\bar{k} \in \mathbb{N}$ and $\bar{B} \in \mathcal{A}$ such that $(x^k, p^k) \in \bar{B}$ for all $k \geq \bar{k}$. In other words, the sequence (x^k, p^k) is eventually contained in one of the sets of continuity of the function $\Psi^x(p)$.

By (5), for fixed i the function $s_i^x(p)$ is continuous with respect to the variables (x, p) in the set

$$A_1 \times \ldots \times [0, L_i) \times \ldots \times A_N \times [p_{\min}, p_0],$$

with $A_j = [0, L_j]$ for all $j \neq i$, since the only set of discontinuity is $\{x_i = L_i\}$. Moreover, the restriction of $s^x(p)$ to the hyperplane $\{x_i = L_i\}$ is continuous by definition, hence $s_i^x(p)$ is continuous with respect to (x, p) in

$$A_1 \times \ldots \times \{L_i\} \times \ldots \times A_N \times [p_{\min}, p_0],$$

with $A_j = [0, L_j]$ for all $j \neq i$. By repeating this argument for each bank and summing up $s(x, p) = \sum_i s_i(s, p)$, we get claim 1.

We can now prove claim 2. Let $x^* \in [0, L_1] \times \ldots \times [0, L_N]$ be $x^* = \lim_k x^k$. For fixed i, either $x_i^* = L_i$ or $x_i^* \in [0, L_i)$. In the first case, since the sequence x^k is non-increasing, then we have $x_i^k = L_i$ for all k, we set $k_i^0 = 0$. In the latter case, by definition of limit there exists $k_i^0 \in \mathbb{N}$ such that $x_i^k < L_i$ for $k \geq k_i^0$. Up to permutations of the indices, we can assume that $x_i^* = L_i$ for $i = 1, \ldots, I$ and that $x_i^* < L_i$ for $i = I + 1, \ldots, N$. Then for all $k \geq \bar{k} := \max_i k_i^0$ we have

$$x^k \in \{L_1\} \times \ldots \times \{L_I\} \times [0, L_{I+1}) \times \ldots \times [0, L_N).$$

There follows that there exist $\bar{k} \in \mathbb{N}$ and $\bar{B} \in \mathcal{A}$ such that $(x^k, p^k) \in \bar{B}$ for all $k \geq \bar{k}$, and $\Psi^x(p)$ is continuous in $\bar{B}$. Then we have $\lim_k \Psi^k(p^k) = \Psi^{x^*}(p^*)$; on the other hand, $\Psi^k(p^k) = p^k \to p^*$ by (19), hence

$$\Psi^{x^*}(p^*) = p^*,$$

as desired. ∎

4 Simulations for the single asset case

In this section we present the results of a few simulation exercises that are meant to illustrate how the model works and how asset fire sales and direct balance-sheet exposures may interact, spreading contagion through the system.[6]

4.1 Simulations set-up and stylized network topologies

The CFS model was tested by the authors on a uniform, totally interconnected network of ten banks, i.e. a banking system in which all banks have equal balance-sheets and lend to or borrow from one another the same amount of money. This

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[6]In this connection, a caveat is in order: the model has a theoretical purpose and is more suitable for a qualitative analysis than for exact calibration to real data. As a consequence, the numerical values assigned to the parameters in the simulations and the results presented have an illustrative aim only and are not meant to provide quantitative estimates of potential losses in real banking systems.

framework is useful to discriminate patterns in the results that are due to intrinsic characteristics of the model from patterns that might emerge as a consequence of specific features of the banking system at hand. Therefore, we will consider this benchmark case as well in our assessment of the model.

On the other hand, the purpose of introducing networks in economic and financial models relates to the need to explicitly account for the structure of interconnections that characterizes the financial system. Indeed, this structure (the topology of the network) may deeply influence the behavior of the system. In particular, it may contribute to the resilience of the financial system to exogenous shocks, or, on the contrary, increase havoc (see the seminal paper by Allen and Gale, 2000). It is then important to consider how the model behaves when performing contagion analysis on different network topologies. To this end, in the following paragraphs we consider four main archetypal topologies (Figure **??**):

(i) *totally interconnected topology* – each bank is linked to all other banks (as described above);

(ii) *circle topology* – each bank borrows from only one bank and only lends to another bank, i.e. bank i borrows from $i - 1$ and lends to $i + 1$, for $i = 1, \ldots, N$ (bank 1 borrows from bank N);

(iii) *star topology* – there is only one big bank (the core of the star) that borrows from and/or lends to all other banks in the system, while non-core banks are not interconnected with one another;

(iv) *core-periphery topology* – similar to the star topology, there is a core of big banks that interact on the interbank market with smaller banks (peripheral banks).

While the first three topologies are highly stylized, the last one represents a situation that is quite common in real banking systems. Though differentiating the way in which banks are interconnected, we assume a uniform structure in terms of balance-sheet dimensions and composition, except for the banks belonging to the core of the star or the core-periphery networks. We consider networks of 100 banks, with the initial (i.e., pre-shock) balance-sheet of a representative bank in the system as in Table 1. In the case of the star topology, we add to the 100 uniform banks a core bank, the total assets of which are five times bigger in value than those of the representative bank. Peripheral banks are assumed to be split in two equinumerous groups, according to whether they lend to or borrow from the core bank. Finally, in the core-periphery network, the core consists of 10 uniform banks, with total assets ten times those of peripheral banks. The latter are divided in 10 groups (of 10 banks each). The banks in each group exchange money in the interbank market between one another (assuming total interconnectedness inside each group) and with one and only one bank belonging to the core. Core banks, similarly, lend to and borrow from any other bank belonging to the core, in addition to peripheral banks belonging to the associated group.
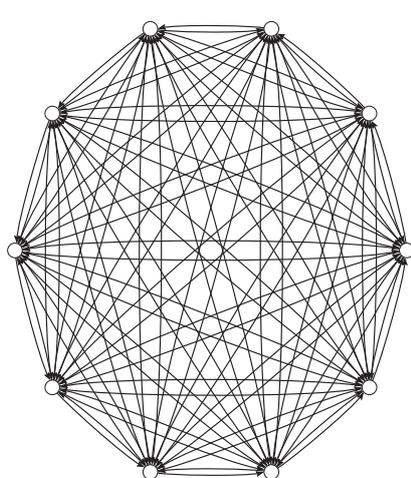
(a) Totally interconnected
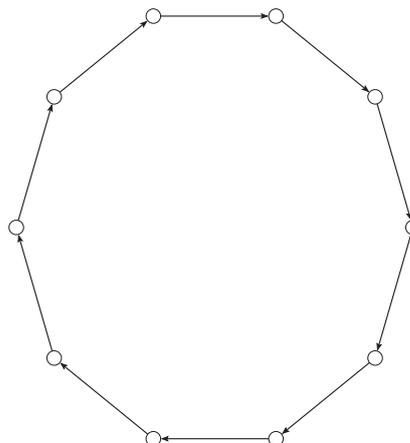
(b) Circle
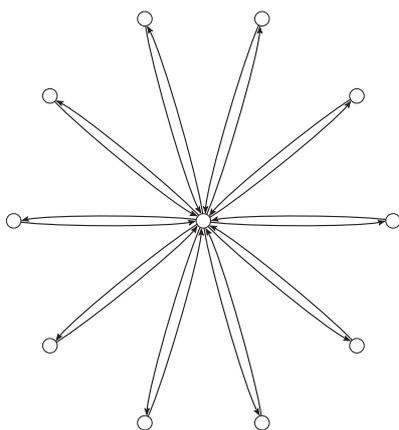
(c) Star
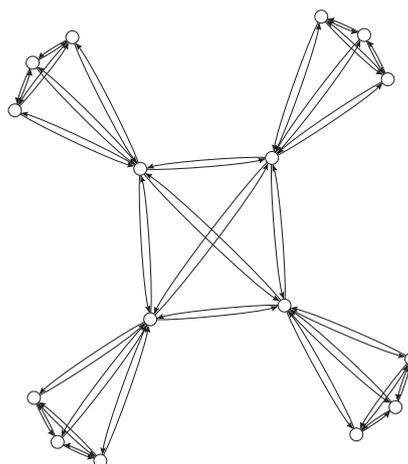
(d) Core-periphery

Figure 1: Stylized network topologies

Table 1: Representative bank's balance-sheet

Assets		Liabilities	
Liquid asset:	$c_i = 40$	Non-interbank liabilities:	$d_i = 160$
Illiquid asset:	$e_i = 130$	Interbank liabilities:	$L_i = 30$
Interbank assets:	$\sum_{j \neq i} x_j \pi_{ji} = 30$		
		Equity:	$K_i = 10$

In the following sections we present four main simulation exercises implemented with this basic set-up. Unless otherwise specified, we set the regulatory leverage ratio equal to 4% (a figure broadly in line with leverage ratios of many commercial banks and with the 3% requirement envisaged by the Basel 3 framework) and the minimum price that can be attained by the illiquid asset equal to 0.9 (the initial price, when no liquidation has occurred yet, is normalized to 1, as the price of the liquid asset).[7] The inverse demand function is assumed to be quadratic and is accordingly fitted to the minimum price as:[8]

$$p(s) = 1 - \alpha s^2, \quad \text{where} \quad \alpha = \frac{1 - p_{min}}{\left(\sum_{j=1}^{N} e_j\right)^2}.$$

This functional form represents the easiest way to instance the assumption of a concave inverse demand function, which better fits the empirical behavior of prices in case of fire sales than a convex function (like the exponential one) does. Indeed, the price of an asset is usually little affected by small sellings, but it may significantly deteriorate, and at a faster pace, in case there are substantial sellings.

Furthermore, in each exercise contagion analysis is performed in a standard way: an exogenous shock hits one or more banks in the system and the ensuing propagation of distress is measured. A bank defaults if, after liquidating all of its non-interbank assets, it does not comply with the regulatory leverage ratio. In this connection, note that:

(i) As is common in the literature, we adopt a "timeless" perspective, according to which contagion spreads instantaneously (there is no time dimension in the model) and banks do not have the possibility to readjust the composition of their assets in response to a shock or to defaults of other banks (they are only allowed to sell as many assets as needed to comply with the regulatory leverage ratio).

(ii) The rate of recovery for investors exposed to a defaulting bank is determined endogenously in the model. As a consequence, the magnitude of the initial shock matters and we do not need to restrict ourselves to consider

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[7]Obviously, these parameters play a key role in determining the outcomes of any simulation exercise (a sensitivity analysis is presented in Section 4.5). In particular, the choice of the minimum price is not straightforward. For instance, in Cifuentes, Ferrucci and Shin (2004), the extended working paper version of the CFS model, in most simulations the authors set the minimum price equal to 0.5. On the other end of the scale, Gauthier, Lehar and Souissi (2010) set it equal to 0.98. While modeling real fire sales would require to pick a low value of the minimum price, the stylized structure of the model, in which only one type of illiquid asset is available on the market, imposes a trade-off: on one hand, low values of the minimum price would produce the unrealistic result that even a modest shock could trigger an endemic contagion yielding to the wipe-out of the whole system; high values, on the other hand, are unrealistic but allow to study a wider range of outcomes assuming reasonable shock values.

[8]Section 4.5 provides an example of how a different assumption on the functional form may affect the results of the simulations.

the complete wipe out of a bank's total assets as an initial trigger of the contagion process.

(iii) We only consider here shocks to the asset side of the balance-sheet.[9] More precisely, the initial shock is modeled as a loss of a given share of the illiquid asset, affecting one or more banks in the system.[10]

(iv) In the CFS model, distress may be transmitted even if no default occurs: as soon as a bank is forced to sell part of the illiquid asset in its portfolio in order to comply with the regulatory ratio, the price of that asset deteriorates and, as a consequence, the value of other banks' illiquid assets diminishes as well.

Finally, when evaluating the impact of fire sales and balance-sheet contagion on the system, we will compute the following metrics:

- the share of liquid assets sold by bank i, $\%Liq = t_i/c_i$;

- the share of illiquid assets sold by bank i, $\%Illiq = s_i/e_i$;

- the share of interbank liabilities that bank i is unable to repay, $\%IB = 1 - x_i/L_i$;

- the negative of the change in total asset value, with respect to the value of assets recorded after the shock hits and before the contagion process starts, namely

$$\Delta_{TA} = 1 - \frac{\sum_{j=1}^{N} \left(c_j - t_j + p \cdot (e_j - s_j) + \Pi' x \right)}{\sum_{j=1}^{N} \left(c_j + p_0 \cdot e_j + \Pi' L \right)},$$

where p_0 and e_j denote respectively the price of the illiquid asset and bank j's endowment of illiquid asset immediately after the initial shock, before any liquidation process starts;

- the (positive) loss of senior creditors (e.g., depositors or bondholders)

$$Loss_D = \frac{\sum_{j=1}^{N} \max\{0, d_j - (c_j + p \cdot e_j + \Pi' x)\}}{\sum_{j=1}^{N} d_j};$$

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[9]The financial crisis has shown that considerable problems to bank balance-sheets may stem from the liabilities side and part of the recent literature has shifted towards the consideration of funding liquidity issues. However, we do not include this extension here in order to avoid complicating the model excessively, before having clarified the structure of its original version (see also the comments in May and Arinaminpathy, 2010, on this topic).

[10]Once the shock hits the system, the price of the illiquid asset is updated, as if the amount of illiquid asset eliminated by the shock were sold on the market. Running simulations without this adjustment, one could draw the paradoxical conclusion that a more severe shock would be better than a lighter one. In fact, a severe shock would leave the hit bank with less illiquid asset available for sale, i.e. with a smaller potential to negatively impact other banks via the fire sales channel (see Section 3.1).

- the number of defaults, i.e. the number of banks $j = 1, \ldots, N$ such that $t_j = c_j$, $s_j = e_j$ and

$$r = \frac{c_j + p \cdot e_j + \Pi'x - d_j - L_j}{\Pi'x} < \bar{r}.$$

4.2 Contagion analysis

The first simulation exercise we present is a standard contagion analysis, performed on the four different stylized topologies described above. In particular, we fix beforehand a set of banks that are hit by an exogenous shock and calculate the impact metrics letting the magnitude of the shock vary. More precisely, the chosen banks are affected by an initial shock eroding a share S of the illiquid asset endowment of each of them, with S ranging between 0 and 100 per cent (for simplicity, S is equal for all banks hit by the exogenous shock). The results are shown in Figure 2.

The first overall conclusion conveyed by the results is the existence of "phase transition" points in each configuration, i.e. threshold values of the initial shock above which the number of defaults shifts from virtually zero to the number of banks hit by the shock, and then to the total number of banks in the system. While exacerbated by the simplicity of the model and by the chosen set-up (homogeneous banks), a regime-switching behavior based on different values of the relevant parameters may be found also in more realistic network models (see e.g. Newman, 2003) and should be taken into account when assessing the resilience of real financial networks.

Considering more in detail the four topologies analyzed, Figure 2 displays a few remarkable differences. As shown by panels (a) and (b), the total interconnected topology and the circle configuration display similar outcomes with respect to all metrics, although the circle starts yielding to the default of all banks at a lower level of the initial shock (roughly 20 per cent, instead of 30). Thus, compared with the totally interconnected topology, it displays a narrower range of shocks leading to non-catastrophic contagion.[11] On the other hand, the transition between the two regimes (either only the banks hit by the shock or all banks default) is more gradual, due to the heterogeneous structure of the interbank exposures.[12] In the case of the star and core-periphery configurations, where heterogeneity between bank balance-sheets matters by definition, the results depend on how the initial shock is targeted. If it only affects peripheral banks (Figure 2, panels (c) and (d)), these topologies prove to be particularly resilient, as contagion effects are not able to trigger additional defaults (apart

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[11]This conclusion does not hold if there is a clustering of the banks hit by the shock. In this case (not shown in Figure 2), the circle configuration may be especially apt to dampen balance-sheet contagion effects, due to the scantly interconnected structure of interbank exposures.

[12]Recall from Section 4.1 that the heterogeneity stems here only from the structure of the unweighted adjacency matrix (i.e., the matrix of interbank liabilities obtained from L replacing all non-zero elements with 1), as all links have equal weights (all banks have equal interbank assets/liabilities).

Figure 2: Contagion analysis and magnitude of the initial shock (1)

(a) *Totally interconnected*

(b) *Circle*

(c) *Star (core not hit by the shock)* (2)

(d) *Core-periphery (core not hit by the shock)* (2)

(e) *Star (core hit by the shock)* (3)

(f) *Core-periphery (core hit by the shock)* (4)
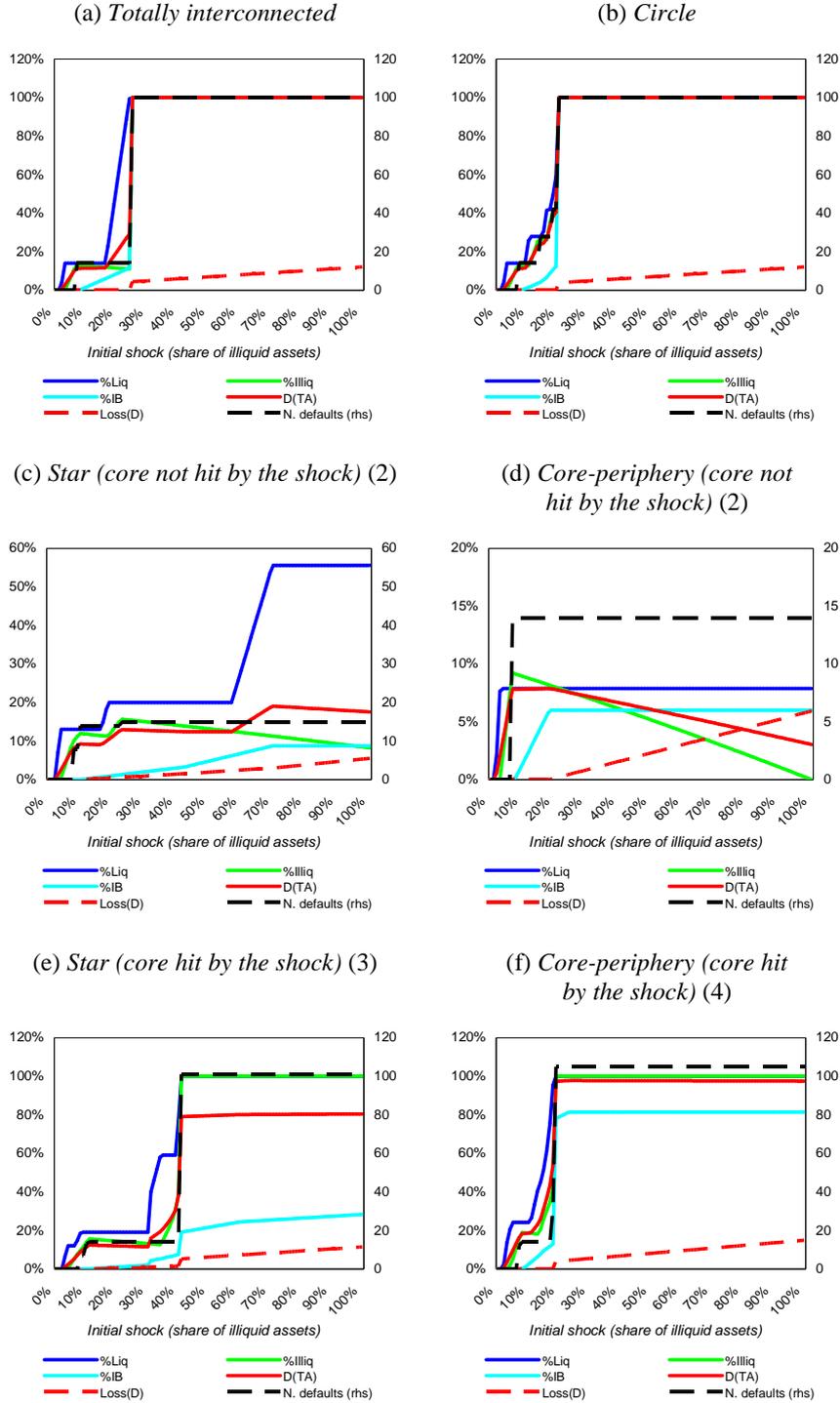
(1) On the x axis the share S of illiquid assets canceled by the initial shock for each affected bank is reported, while the y axis refers to the metrics defined in Section 4.1. The number of defaults (*N. defaults*) is on the right-hand scale. The main parameters of the model are set as follows: $\bar{r} = 4\%$; $p_{min} = 0.9$. – (2) The banks hit by the initial shock are uniformly chosen among the peripheral ones. – (3) The core of the star is affected, in addition to thirteen peripheral banks. – (4) Two core banks are affected, in addition to twelve uniformly chosen peripheral banks.

from those of the banks hit by the shock). On the other hand, as one or more core banks are involved in the set of shocked institutions, the results become very similar to those recorded for the totally interconnected and circle topologies, with the only difference that, even in the regime in which the whole banking system defaults, the interbank assets are not completely wiped out (about 30 and 80 per cent of interbank loans are not paid back in the star and core-periphery topologies, respectively). Moreover, while the cut-off is similar for the core-periphery and circle configurations (an initial loss of about 20 per cent of illiquid assets of the banks hit by the shock), it is far larger for the star topology (above 40 per cent), suggesting that the overall resilience of a core-periphery type of network depends substantially on the relative dimension of the core, as compared with the periphery.

In conclusion, the results of this exercise indicate that a star or core-periphery topology might be better than other stylized network configurations in order to dampen contagion effects, as implied also by current regulatory effort to bring securities markets transactions to be cleared by central counterparties (CCPs). However, the resilience of these topologies may deteriorate significantly when they are subject to targeted shocks hitting the core of the network, especially if the dimension of the core is huge compared with that of peripheral banks. This calls for additional care by supervisors in controlling risk management practices at core banks (or CCPs, extrapolating the results to the case of securities markets). Moreover, it may justify the introduction of additional capital buffers for core institutions, as envisaged by the Financial Stability Board (FSB) for systemically important financial institutions (SIFIs). On the other hand, by the same argument one could advocate lower capital buffers for peripheral banks, compared with banks having similar asset size but organized according to different network structures. In other words, the "interconnectedness" of a bank is not equivalent to the size of its total interbank assets and liabilities, but is intrinsically determined by the actual decomposition of those aggregate figures at the level of bilateral exposures (i.e., the topology of the network).

4.3 Capital injections: An allocation problem

In this section we show how our framework can be used to deal with a problem of optimal recapitalization of a banking system subject to budget constraints. Suppose that a huge shock hits all the banks and that the Government wants to inject new capital in the system, but it is able or willing to do so only up to a certain point (there remains an unfilled gap of M per cent of the pre-shock illiquid asset endowment of the banking system). In this context, how should the Government allocate its constrained resources in order to restore the initial conditions of the banking network? Arguably, given that bank balance-sheets in our stylized networks are homogeneous, it should simply inject equal quantities of capital in each bank. However, we can show that if the shock exceeds a certain threshold then such a uniform capitalization is not optimal and a diversified strategy leads to a preferable outcome, in terms of both number of defaults and

total assets' value deterioration.

For notational simplicity we will first rephrase the problem from a reverse perspective and then we will interpret the results in terms of optimal recapitalization. Assume that the system is hit by a shock which cancels overall a percentage M of the total illiquid assets; we want to analyze to what extent the final impact depends on how the shock is distributed among several banks. To this end we split the shock in equal proportions among an increasing number n of banks ($n = 2, \ldots, 100$). In particular, we choose the banks to be hit by the shock in such a way that the serial numbers identifying the n banks are equally spaced over the range $1, \ldots, 100$ (e.g., if $n = 5$, we can shock banks 1, 21, 41, 61 and 81) and, in the case of the star and core-periphery topologies, no core banks are hit by the initial shock.[13]

The results are shown in Figure 3, where the impact in terms of number of defaults (panels $(a1)$, $(b1)$, $(c1)$ and $(d1)$) and change in total asset value (Δ_{TA}; panels $(a2)$, $(b2)$, $(c2)$ and $(d2)$) are plotted against the percentage M of initial shock (x axis; $M = 1\%, \ldots, 4\%$) and the number n of banks hit by the shock (y axis; $N = 2, \ldots, 100$), for all four stylized network configurations.

A common pattern emerges from these plots, indicating a regime-shifting behavior. Irrespective of the topology and the chosen distress metric (either number of defaults or change in total asset value), there exists a threshold value $\overline{M}$ of the initial shock with the following characteristics:

- If $M \leq \overline{M}$, there exists a threshold $\overline{n}(M)$ such that the impact of contagion is increasing (or non monotone, in the case of the circle topology) in the number n of banks hit by the shock if $n \leq \overline{n}$, decreasing otherwise (in particular, the number of defaults abruptly goes down to zero). This means that, if the initial aggregate shock is not too big, it may be better born by the system if it is split among a sufficient number of banks, as in this case the initial loss per bank can be easily absorbed selling liquid assets (or an amount of illiquid assets not so substantial as would be necessary to trigger a fire sale).

- If $M > \overline{M}$, there exists a threshold $\overline{n}(M)$ such that the impact of contagion is increasing (or non monotone, in the case of the circle and star topology) in the number n of banks hit by the shock if $n \leq \overline{n}$, constantly equal to the maximum level of distress reached at $n = \overline{n}$ otherwise (all banks default). In this case, the magnitude of the initial shock is so large that its impact is less material if the shock is split among a few banks than if it is spread over the whole system. Indeed, in the latter case the initial loss faced by each bank is sufficiently big to force the bank to sell all of its assets without being able to comply with the regulatory ratio.

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[13]This is consistent with the goal of this exercise, i.e. an optimal recapitalization problem: we saw in the previous section that when the core takes a loss, the impact on the star and core-periphery networks is more substantial than if only peripheral banks are hit; in this case, then, the choice of how to allocate capital may be more straightforward.

Figure 3: Contagion analysis and distribution of the initial shock (1)

(a1) *Totally interconnected − N. defaults*   (a2) *Totally interconnected − Loss$_{TA}$*
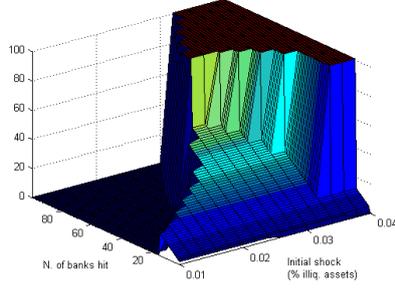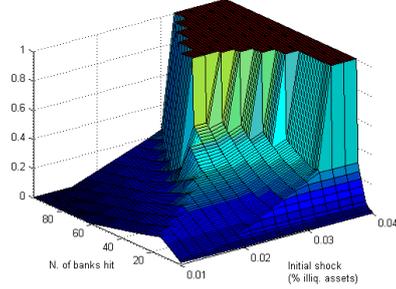

(b1) *Circle − N. defaults*   (b2) *Circle − Loss$_{TA}$*
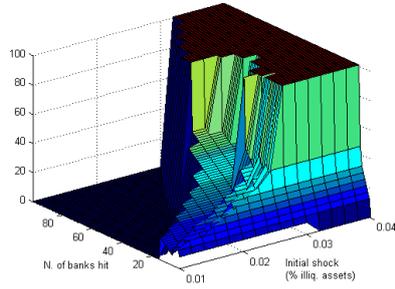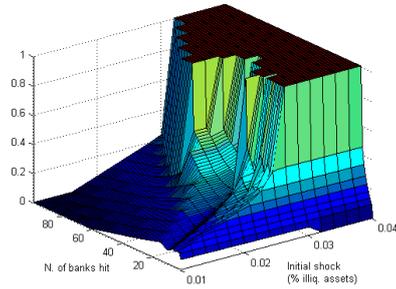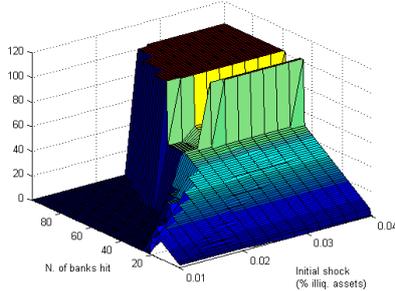
(c1) *Star − N. defaults*   (c2) *Star − Loss$_{TA}$*
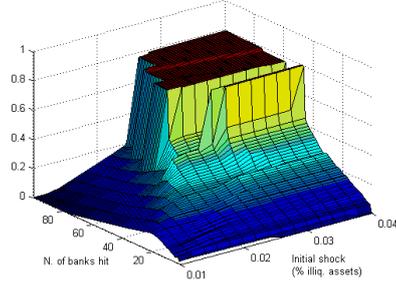
(d1) *Core-periphery − N. defaults*   (d2) *Core-periphery − Loss$_{TA}$*
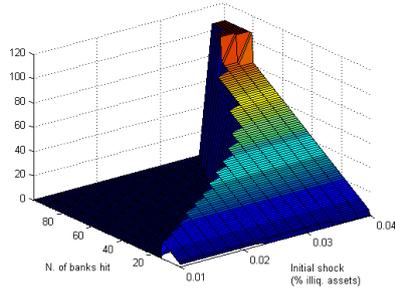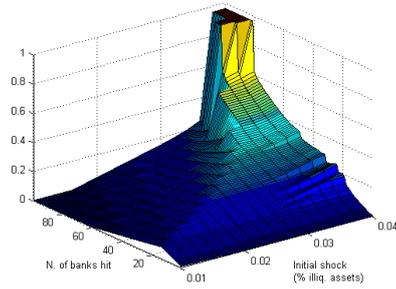
(1) On the x axis, the aggregate magnitude M of the initial shock is reported ($M = 1\%, \ldots, 4\%$ of aggregate illiquid asset endowment); on the y axis, the number n of banks hit by the initial shock ($n = 2, \ldots, 100$); on the z axis, either the number of defaults ($N.defaults$) or the change in the value of assets (Δ_{TA}), as defined in Section 4.1. The main parameters of the model are set as follows: $\overline{r} = 4\%$; $p_{min} = 0.9$. The banks hit by the initial shock are uniformly chosen in the system (among the peripheral ones only, in the star and core-periphery configurations).

While the first part of the previous result can be intuitively grasped in terms of "diversification" (the more spread out the shock, the less substantial its impact), the second part indicates that, if the aggregate shock is big enough, dilution of the shock among many banks is not beneficial for the network. This remark suggests that, when coping with systemic risk, not only the magnitude of the shock should be taken into account, but also how it is distributed across the system, even in the stylized case of homogeneous banks.

The policy implication of this conclusion may be better rephrased if we interpret the exercise reported in Figure 3 as a problem of optimal recapitalization of the banking system subject to a budget constraint. We assume that a huge shock hits all the banks and the Government wants to inject new capital in the system (or force banks to raise additional capital),[14] but it is able or willing to do so only up to a certain point: here M is the percentage of pre-shock illiquid asset endowment which remains unfilled after the recapitalization. In this context, how should the Government allocate its constrained resources in order to restore the initial conditions of the banking network? Arguably, given that bank balance-sheets in our stylized networks are homogeneous, it should simply inject equal quantities of capital in each bank. However, Figure 3 proves that, whenever the initial shock is greater than a given threshold value $\overline{M}$, such a uniform recapitalization would not be optimal. A uniform recapitalization of all banks except a few of them would do better, both in terms of number of defaults and total assets' value deterioration.

4.4 The incremental effect of fire sales

The results of the previous exercises did not allow to discern the relative importance of the two different channels of contagion considered in the model. To this end, we perform two contagion analysis exercises with the same initial shock and parameters, except that in one of them we consider all assets as liquid (this simply amounts to set $p_{min} = 1$).[15] In this case, direct balance-sheet exposures are the only possible channel of contagion, as no fire sale effects can emerge. Different values of the metrics described in Section 4.1 across the two exercises are then a proxy of the incremental effect obtained assuming that one asset is illiquid.

In particular, Figure 4 shows how the change in total assets' value Δ_{TA} changes as an initial shock of fixed magnitude is split among an increasing number of banks (similar to the exercise presented in Section 4.4, but with the aggregate magnitude M of the initial shock alternatively set equal to two different given values, for the sake of simplicity). In each panel, the red line refers to the case with fire sales, while the blue line represents the case with liquid assets only.

The results of these simulations suggest that the effect of asset fire sales adds to

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[14]To be consistent with the model, assume that the capital injections take place immediately after the shock, before marking-to-market the illiquid asset.

[15]This analysis is a particular case of the sensitivity analysis presented in Section 4.5, but is presented separately in order to explicitly focus on the effect of fire sales per se, independently on the degree of illiquidity assumed.

Figure 4: Incremental effect of illiquidity (1)
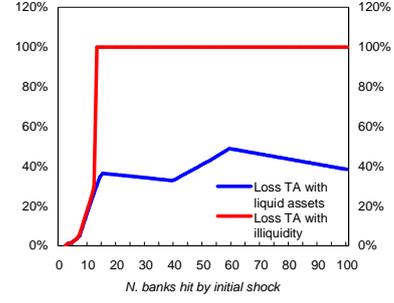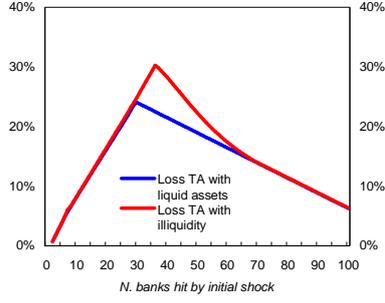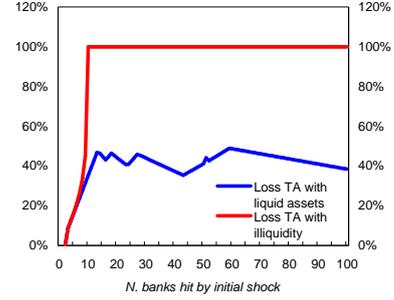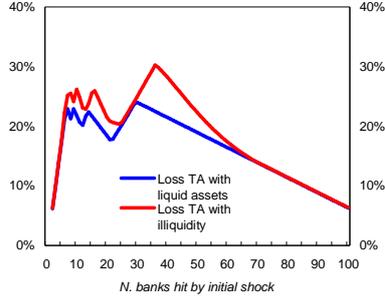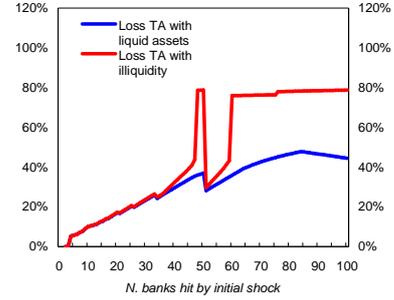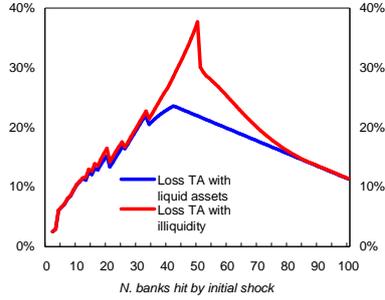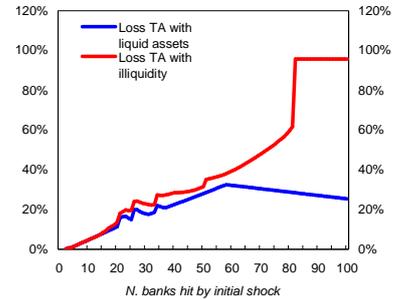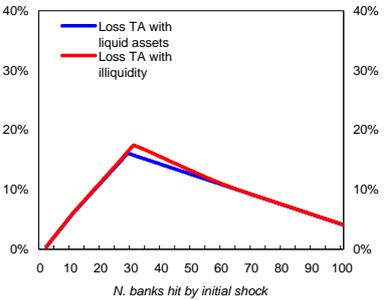
(1) On the x axis, the number n of banks hit by the initial shock ($n = 2, \ldots, 100$) is reported (the aggregate magnitude M of the initial shock equals either 2% or 4% of the aggregate illiquid asset endowment); on the y axis, the change in the value of assets is reported, as defined in Section 4.1. The main parameters of the model are set as follows: $\bar{r} = 4\%$; $p_{min} = 0.9$. The banks hit by the initial shock are uniformly chosen in the system (among the peripheral ones only, in the star and core-periphery configurations).

the direct balance-sheet exposure channel of contagion in an incremental way and then gradually fades away as the number n of banks hit by the initial contagion tends to 100 (the total number of banks in the system – the number of peripheral banks, in the star and core-periphery topologies). Indeed, in each panel of Figure 4 three main intervals of values of n may be broadly discerned, that merge with one another in a continuous way:

- For n sufficiently small (independently of the value of M), the shock is confined to a few banks in the system and can not trigger enough illiquid assets sales to generate a sizable price deterioration. In this case, direct balance-sheet exposures (loans on the interbank market) are the main channel of contagion and the final impact of the exogenous shock can hardly be distinguished in the cases with or without illiquid assets.

- As n takes on "intermediate" values (the exact range depends on the network topology and the value of M), more banks have to liquidate their illiquid asset endowment, generating a material effect on price. As a consequence, the final impact on the total assets' value is significantly different from that recorded in the case with liquid assets only. However, as n is in this range, the difference is mainly in levels, while in each panel the two lines show qualitatively similar contours overall.

- Finally, as n takes sufficiently high values, depending on the magnitude M of the initial shock there may be two completely different outcomes. If M is low (2% of the aggregate illiquid asset endowments in Figure 4, panels $(a1)$, $(b1)$, $(c1)$, $(d1)$), then the exogenous shock is split among a sufficiently high number of banks. Therefore, the shock faced by each bank is relatively small and the regulatory leverage ratio can be met without selling substantial amounts of illiquid asset. In this case, the final impact (as measured by Δ_{TA}) displays negligible differences (if any) in the case with the illiquid asset and the case with liquid assets only. On the contrary, if M is high (4% in the figure, panels $(a2)$, $(b2)$, $(c2)$, $(d2)$), all banks in the system[16] eventually lose their assets in full due to the effect of fire sales (differently from the case with no liquid assets).[17]

4.5 Sensitivity analysis: Relative effectiveness of different policies

Finally, we present the results of an analysis of sensitivity of the model to its key parameters, i.e. the leverage ratio $\bar{r}$ that the banks have to comply with and the minimum price p_{min} attainable in case all banks sell their illiquid asset endowments in full. This is equivalent to investigating how the resilience of the banking system depends on its initial capitalization level and on the degree of

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[16]Peripheral banks only, in the case of the star or core-periphery topologies.

[17]For higher values of M (not shown in Figure 4), the shock may be huge enough to trigger the default of all banks also in the case with liquid assets only.

illiquidity of the asset with non-zero price elasticity. In addition, we assess how the functional form of the inverse demand function (exponential or polynomial) may affect the contagion analysis results.

The relevance of this exercise is given by its policy implications. In the simplified world described by the model, a policymaker could act on two levers to make the banking system more resilient: (i) require banks to hold more liquid assets in their portfolios; (ii) ask for higher capital levels. We will show that the two instruments may not be perfect substitutes, especially for some network topologies. Moreover, we will show that the relative importance of the two policy intruments may also depend on the characteristics of the initial shock and on the properties of the demand of illiquid assets.

The results of this simulation excercise are shown in Figure 5: we let $\bar{r}$ range between 2% and 5% (x axis) and p_{min} over the interval $[0.2, 1.0]$ (y axis), and plot against them the deleveraging of liquid assets ($\%Liq$; panels $(a1)$, $(b1)$ and $(c1)$) and the change of assets' value (Δ_{TA}; panels $(a2)$, $(b2)$ and $(c2)$).[18] As expected, the final impact of the shock (i.e., the role played by contagion) depends heavily on how the parameters are initialized: an initial shock of fixed magnitude can either trigger no deleveraging (if both assets are liquid and the regulatory leverage ratio is low) or force all banks to sell the liquid assets in their portfolios and eventually default.

However, the sensitivity analysis also shows how the relative importance of the two parameters in triggering contagion depends on the topology considered, something that could not be easily predicted ex ante. In particular, in the totally interconnected configuration (Figure 5, panels $(a1)$, $(a2)$) the level of the regulatory ratio influences the final impact of the shock significantly more than the minimum price does. Irrespective of how low (high) the price of the illiquid asset can be, no deleveraging occurs (all banks default, respectively) if the regulatory ratio is sufficiently low (high). In the circle topology (panels $(b1)$, $(b2)$), on the contrary, the elasticity of the price of the illiquid asset is the leading parameter. No matter how highly capitalized the banking system is (i.e., how low the regulatory ratio is, compared with the actual leverage of the banks), if the asset is sufficiently illiquid, all banks will eventually go bankrupt. Finally, the star topology (panels $(c1)$, $(c2)$) is somewhere in the middle between these two extreme situations. Indeed, there is no clear hierarchy between the two parameters: low leverage (i.e., a high value of the ratio r) and high liquidity of the assets are equally relevant for the soundness of the banking system.

The previous remarks give an insight on the relative importance of the two instruments which can be used by a policymaker to make this stylized banking system more resilient (require banks to hold more liquid assets in their portfolios, or ask for higher capital levels). We have seen that, depending on the topology of the banking network, the two instruments may not be perfect substitutes.

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[18]In order to better visualize the qualitative behavior of the model as a function of the parameters, the exercise reported in Figure 5 is performed assigning shocks of different magnitude to each stylized network configuration (results for the core-periphery topology are similar to those of the star topology and are omitted).

Figure 5: Sensitivity to regulatory ratio and illiquidity (1)

(a1) *Totally interconnected − %Liq*
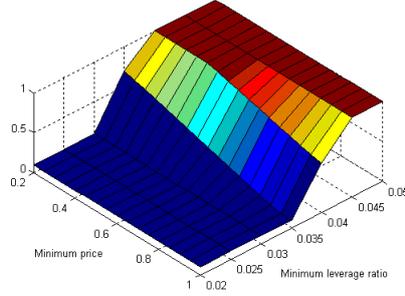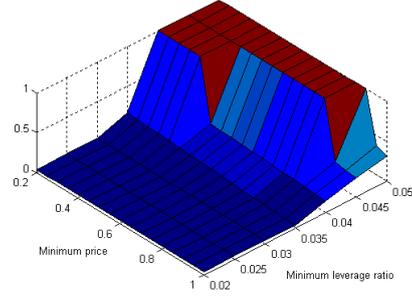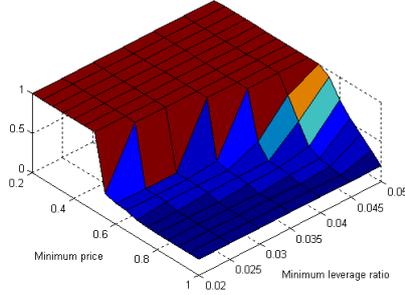
(a2) *Totally interconnected − Loss$_{TA}$*

(b1) *Circle − %Liq*

(b2) *Circle − Loss$_{TA}$*
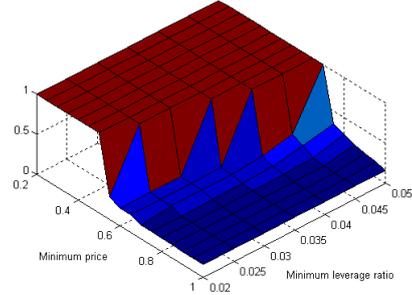
(c1) *Star − %Liq*
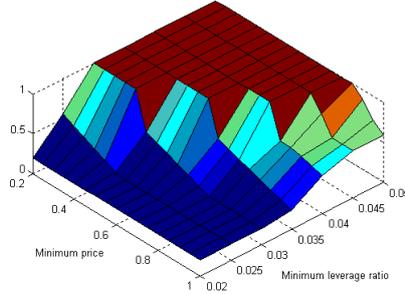
(c2) *Star − Loss$_{TA}$*
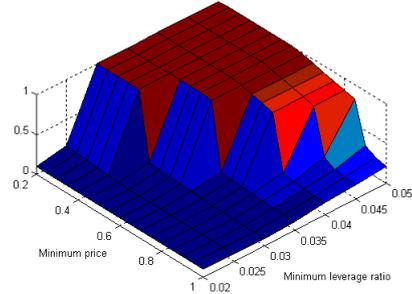
(1) On the x axis, the regulatory leverage ratio $\overline{r}$ is reported; on the y axis, the minimum price (p_{min}) that can be attained by the illiquid asset; on the z axis, either the deleveraging of liquid assets (%Liq) or the change of the value of assets (Δ_{TA}), as defined in Section 4.1. The initial shock consists of the wipe-out of illiquid assets of n banks ($n = 10$ in the totally interconnected topology; $n = 5$ in the circle; $n = 14$ in the star). The banks hit by the initial shock are uniformly chosen in the system (among the peripheral ones only, in the star topology).

Figure 6: Sensitivity to regulatory ratio and illiquidity in the case of an exponential inverse demand function (1)

(a1) *Totally interconnected − %Liq*   (a2) *Totally interconnected − Loss_{TA}*
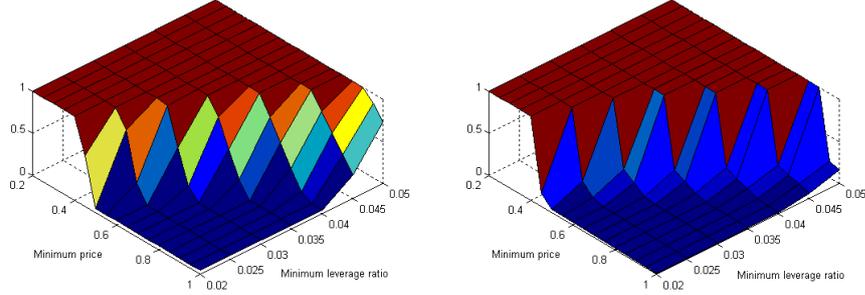

(b1) *Circle − %Liq*   (b2) *Circle − Loss_{TA}*
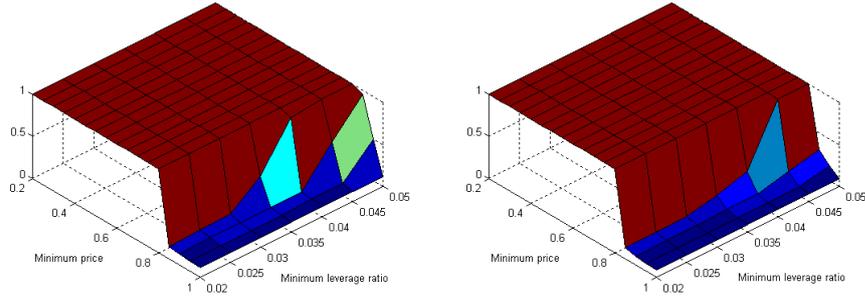

(c1) *Star − %Liq*   (c2) *Star − Loss_{TA}*
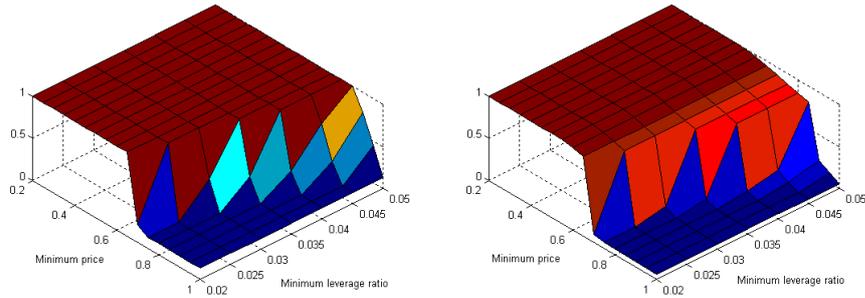
(1) On the x axis, the regulatory leverage ratio $\bar{r}$ is reported; on the y axis, the minimum price (p_{min}) that can be attained by the illiquid asset; on the z axis, either the deleveraging of liquid assets (%Liq) or the change of the value of assets (Δ_{TA}), as defined in Section 4.1. The initial shock consists of the complete wipe-out of illiquid assets of n banks ($n = 5$ in the totally interconnected topology; $n = 3$ in the circle; $n = 7$ in the star). The banks hit by the initial shock are uniformly chosen in the system (among the peripheral ones only, in the star topology).

With respect to the specification adopted in the present exercise, while they are substitutes in the star configuration, in the totally interconnected topology it may be useless to try to improve the soundness of the banking system simply increasing the liquidity of banks' assets without acting on capitalization levels (the converse holds for the circle topology).

Finally, the simulations show that the relative substitutability of the two policy instruments may also depend on the characteristics of the initial shock and on the functional form of the inverse demand function.

Obviously, an exponential function implies a more sizable impact of fire sales than a polynomial function does. Therefore, when performing the previous sensitivity analysis under the assumption that the inverse demand function has an exponential form, we have to lower the magnitude of the initial shock substantially in order to obtain results that are qualitatively similar to (or at least comparable with) those discussed above. The results of this exercise with an exponential function are presented in Figure 6, where the magnitude of the initial shock is roughly half of that assumed in Figure 5. It is apparent that the relative importance of the minimum price over the regulatory leverage ratio is greater in the exponential case, thus shifting the balance between the two policy instruments considered. Although the results for the circle topology are basically unchanged (the supremacy of liquidity conditions is confirmed), the totally interconnected topology now displays a higher degree of substitutability between the two instruments (similar to the star configuration in Figure 5), while the behavior of the star topology with respect to changes in the two parameters of the model becomes closer to that of the circle configuration.

5   An extension to multiple illiquid assets

The CFS model assumes a stylized representation of a bank's balance-sheet, in which there are only two types of assets (apart from the interbank loans), i.e. a totally liquid asset and an illiquid one. A natural generalization of the model that would make it more realistic consists of expanding the range of available assets.[19]

In the context of the CFS model, there are only two features that can characterize an asset: (i) its degree of liquidity (described by the inverse demand function); (ii) how it is distributed across banks' portfolios. The interaction between these two factors determines to what extent a bank can contribute to or is affected by price deterioration in the case of fire sales. Analogously, when considering several assets, we can discriminate among them assuming different degrees of liquidity and/or different endowments across banks.

The question then arises as to how a bank should liquidate its assets when forced to deleverage in order to comply with the regulatory ratio (for a given

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[19]Gauthier, Lehar, Souissi (2010) partly move in this direction, introducing heterogeneity in prices of different banks' illiquid assets. However, they model this as a fluctuation around the fundamental price derived in the CFS framework, due to heterogeneous levels of riskiness of individual banks. On the contrary, we want to model assets that may differ per se in their degree of liquidity and have the same value for different banks holding them.

interbank payments vector x). In the single illiquid asset case considered in the CFS model, the authors assumed that the liquidation strategy was the same for all banks: sell as many units of the liquid assets as needed to restore the minimum leverage ratio; if selling the whole endowment of liquid asset does not suffice, then start selling as many units of the illiquid asset as necessary. While this behavioral rule is economically sensible, it does not have analytical underpinnings in the CFS model and is unclear how to generalize it to the multiple assets case that we are considering now.[20] Even though we could agree that a sensible strategy consists of selling more liquid assets first, it is debatable how liquidity should be defined in this context. For instance, at least two strategies can be devised (each prevalently hinging on one of the two aspects that characterize illiquid assets) that may lead to different liquidation decisions: (i) selling first the asset whose price deteriorates less rapidly (e.g., in the case of the exponential inverse demand function in equation (4), selling the asset with a smaller α first); (ii) selling the asset whose minimum price is higher (this depends on the aggregate endowment of illiquid assets of each class, in addition to the parameter α that identifies that class). Notice that, in the case of a liquid asset and an illiquid one only, both strategies reduce to the behavioral rule assumed in the CFS model. In the general case, however, there may be cases in which both of them are sub-optimal, as shown in the following. Thus, the extension of the CFS model to the multiple assets case may add economic insight to the model.

To answer the question on liquidation strategies, we provide a simple microfoundation based on the assumption that each bank is willing to pick any liquidation strategy that maximizes the final value of its equity (equivalent to maximize total assets' value in this simplified setting). In Section 5.1 we present the problem and derive some analytical conditions. In Section 5.2 we show the results of a numerical example.

5.1   Microfoundation of liquidation strategies

In this section we propose a possible way to extend the algorithm described in Section 3.2 to the case of multiple assets. The notation remains broadly unchanged, except that now there are $M + 1$ assets in the economy and we denote by:[21]

- e the *matrix of endowments* – e_{ik} (or e_i^k, as we will write it) is the quantity of asset j that bank i is initially endowed with, $i = 1, \ldots, N$, $k = 0, \ldots, M$;

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[20]In a similar context, Geertsema (2011) assumes that the liquidation is performed on a *pro rata* basis, advocating this choice based on the claim that the results are reasonably similar to those obtain with a sequential liquidation, a strategy consistently employed by many practitioners. We show in the following that both these strategies may be sub-optimal.

[21]For simplicity, one can think of asset 0 as the liquid asset, but in the framework that we are describing it is neither necessary to assume that there is a perfectly liquid asset, nor that it is unique. Therefore, we no longer use a different variable (c or t, respectively) to denote the endowment or selling of the liquid asset.

- s the *matrix of liquidation strategies* – s_{ik} (or s_i^k) is the quantity of asset j sold by bank i to try to comply with the regulatory leverage ratio, $i = 1, \ldots, N$, $k = 0, \ldots, M$;

- p_0 and p_{min} the vectors of after-shock prices and minimum attainable prices, respectively – we assume $0 < p_{min}^k \leq p_0^k \leq 1$ for $k = 0, \ldots, M$;

- $g_k^\sigma : [0, \sum_{i=1}^N e_i^k] \longrightarrow [p_{min}^k, p_0^k]$ the *inverse demand function* of asset $k = 0, \ldots, M$ after a shock σ (in the following, we will drop the index σ) – we assume it to be either constant (if asset k is completely liquid) or continuous and decreasing.[22]

We also denote by e_i, s_i the endowment and strategy vectors of bank i and by e_{-i}, s_{-i} the correspondent matrices of all other banks. Similarly, e^k and s^k will denote the vectors of quantities of asset k that each bank is endowed with or decides to sell according to its liquidation strategy. Furthermore, when focusing on bank i's problem, we will write $g_k(s_i^k, s_{-i}^k)$ instead of $g_k(\sum_{j=1}^N s_j^k)$.

Recall that at each step h of the algorithm, two sub-algorithms have to be implemented, that in the current notation read: (i) given the vector of interbank payments x^{h-1}, find an equilibrium matrix of liquidation strategies s^h and a corresponding price vector p^h ($p_k^h = g_k(s^k)$);[23] (ii) given p^h and s^h, compute the new interbank clearing vector x^h with the adapted version of the fictitious default algorithm by Eisenberg and Noe (2001) described in Section 3.2. The second sub-algorithm is not affected by the extension to the multiple assets case. Therefore, we can focus on the first sub-algorithm only: given a clearing vector x, find an equilibrium liquidation strategy s and the associated price vector p (we drop the h indices for ease of notation).

A standard way to model banks' liquidation strategies is to assume that they try to maximize the final value of their equity. However, this simple assumption brings two issues along. First, a priori the optimal strategy may not be unique. Second, we have to model in which way prices are updated by the bank when evaluating the final value of equity.

As far as the former issue is concerned, one can complement this simple optimization problem with an auxiliary optimization or behavioral rule that discriminates among the optimal solutions found according to a given criterion. As for the latter issue, in the CFS model the behavioral rule adopted does not leave room for any choice on the part of the liquidating bank. As a consequence, it is ultimately immaterial whether the bank considers the price impact of its selling

---

[22]The strict monotonicity assumption (satisfied both by the exponential inverse demand funtion used in the CFS model and by the quadratic function used in most simulations of Section 4) makes the proofs in this section easier. Although it rules out the interesting case of an asset that is liquid up to a certain level of sellings and then becomes illiquid, this case can be easily approximated by a decreasing function with arbitrary small (in absolute value) derivative in the range of sales in which the asset is liquid.

[23]While in Section 3.2 at each iteration bank i's strategy s_i was determined on the basis of the equilibrium price, here it is more convenient to reason in the opposite way, letting the equilibrium price be determined by the interaction of banks' optimal strategies.

or not. However, in the present context the price impact of different liquidation strategies matters and should be taken into account when computing the final value of assets.Therefore, we make the following assumption.[24]

Assumption 5.1 *Every bank i, given the strategies s_{-i} of other banks, picks its optimal strategy $\bar{s}_i$ in the solution set of the problem*

$$
\max_{s_i \, : \, s_i^k \in [0, e_i^k]} \quad \max\left\{0, \sum_{k=0}^{M}(e_i^k - s_i^k)g_k(s_i^k, s_{-i}^k) + (\Pi'x)_i - d_i - L_i\right\}
$$

$$
\text{sub} \quad \bar{r}\left(\sum_{k=0}^{M}(e_i^k - s_i^k)g_k(s_i^k, s_{-i}^k) + (\Pi'x)_i\right) \leq \tag{20}
$$
$$
\leq \sum_{k=0}^{M}e_i^k g_k(s_i^k, s_{-i}^k) + (\Pi'x)_i - d_i - L_i.
$$

The strategy adopted is then

$$
s_i = \left\{ \begin{array}{ll} e_i & \text{if } x_i < L_i, \text{ or problem (20) has no solution} \\ \bar{s}_i & \text{if } x_i = L_i. \end{array} \right. \tag{21}
$$

Taking into account that the minimum ratio requirement does not allow the equity value to become negative and that the previous constraint is always binding at an optimal solution, we can prove that problem (20) can be simplified and replaced with problem (22).

Lemma 5.2 *If, given s_{-i}, the strategy $s_i = (0, \ldots, 0)'$ is feasible, then it is the unique optimal solution to problem (20). Otherwise, under suitable constraint qualification conditions, that problem is equivalent to the following*

$$
\max_{s_i \, : \, s_i^k \in [0, e_i^k]} \sum_{k=0}^{M}e_i^k g_k(s_i^k, s_{-i}^k) \tag{22}
$$

$$
\text{sub} \quad \bar{r}f_i(s_i, s_{-i}) - \sum_{k=0}^{M}e_i^k g_k(s_i^k, s_{-i}^k) + d_i + L_i - (1 - \bar{r})(\Pi'x)_i = 0,
$$

where

$$
f_i(s_i, s_{-i}) = \sum_{k=0}^{M}(e_i^k - s_i^k)g_k(s_i^k, s_{-i}^k).
$$

---

[24]Actually, to ensure that the algorithm presented in Section 3.2 converges, we need an increasing sequence of liquidation strategy vectors $\{s_i^{(h)}\}$, $i = 1, \ldots, N$. Therefore, the box constraints $[0, e_i^k]$ in the optimization problem should be replaced by $[(s_i^k)^{(h)}, e_i^k]$. This would not change the results that follow, but would make the notation more cumbersome. Analogously, we solve the algorithm iteratively, considering in the optimization problem of bank i at step h the liquidation strategies of other banks at step $h - 1$, $s_{-i}^{(h-1)}$, but we drop the indices for simplicity.

Proof. Obviously, the objective function in problem (20) can be replaced by $f_i(s_i, s_{-i})$. Rearranging the constraint, we obtain the equivalent problem

$$\max_{s_i \,:\, s_i^k \in [0, e_i^k]} f_i(s_i, s_{-i})$$

(23)

$$\text{sub } \bar{r} f_i(s_i, s_{-i}) - \sum_{k=0}^{M} e_i^k g_k(s_i^k, s_{-i}^k) + d_i + L_i - (1 - \bar{r})(\Pi' x)_i \leq 0.$$

First of all, as $g_k(\cdot, s_{-i}^k)$ is decreasing and positive, then

$$\frac{\partial f_i}{\partial s_i^k}(s_i, s_{-i}) = (e_i^k - s_i^k) \frac{\partial g_k}{\partial s_i^k}(s_i^k, s_{-i}^k) - g_k(s_i^k, s_{-i}^k) < 0. \qquad (24)$$

Therefore, whenever the strategy $s_i = (0, \ldots, 0)'$ is feasible, it is the unique solution to the problem.

Suppose then that it is not feasible and let s_i be an optimal solution with $s_i^m > 0$ for some $m \in \{0, \ldots, M\}$. The Lagrangian of problem (23) reads

$$\mathcal{L}_i = f_i(s_i, s_{-i}) + \sum_{k=0}^{M} \lambda_k (e_i^k - s_i^k) - \mu \, h_i(s_i, s_{-i})$$

with

$$h_i(s_i, s_{-i}) = \bar{r} f_i(s_i, s_{-i}) - \sum_{k=0}^{M} e_i^k g_k(s_i^k, s_{-i}^k) + d_i + L_i - (1 - \bar{r})(\Pi' x)_i \qquad (25)$$

and $\mu \geq 0$, $\lambda_k \geq 0$ for $k = 0, \ldots, M$. If $\mu = 0$, first order conditions would imply

$$\frac{\partial \mathcal{L}_i}{\partial s_i^m} = \frac{\partial f_i}{\partial s_i^m}(s_i, s_{-i}) - \lambda_m = 0,$$

and, by equation (24),

$$0 \leq \lambda_m = \frac{\partial f_i}{\partial s_i^m}(s_i, s_{-i}) < 0,$$

a contradiction. Therefore, the constraint is always binding at a non-zero solution and can be replaced by the equality constraint. From the new constraint we obtain

$$f_i(s_i, s_{-i}) = \frac{1}{\bar{r}} \Big(\sum_{k=0}^{M} e_i^k g_k(s_i^k, s_{-i}^k) + (1 - \bar{r})(\Pi' x)_i - d_i - L_i \Big).$$

Substituting for the objective function in problem (23) and dropping the constant terms, we obtain problem (22). ∎

Remark 5.3 In the following we will assume that the constraint qualification conditions required for Lemma 5.2 to hold are satisfied. For instance, if the inverse demand functions g_k are concave and C^2 (e.g., quadratic, as we assumed in

Section 4), it is easily verified that the function h_i is (strictly) convex. Therefore all constraints are convex (the box constraints are affine) and Slater qualification condition only requires that there exist a feasible point at which the constraints are not binding. Notice that, under these assumptions, KKT conditions for problem (22) with the equality constraint $h_i = 0$ (and h_i is defined as in equation (25)) replaced by $h_i \leq 0$ are necessary and sufficient conditions for a point to be a local maximum, as all constraints are convex and the objective function is concave (it can be easily shown along the same lines of Lemma 5.2 that the inequality constraint $h_i \leq 0$ is binding at any optimal solution, so that the solution set of problem (22) remains the same if we replace $h_i = 0$ with $h_i \leq 0$).

As a consequence of Lemma 5.2, in the remainder of this section we will focus on problem (22), instead of problem (20). In particular, we derive from the first order condition a few equations that allow us to better understand how optimal liquidation strategies relate to the heuristic strategies discussed above. First of all, we prove that, whenever a bank is endowed with a liquid asset, it is not optimal to sell any amount of illiquid assets, unless the liquid asset has been sold in full. This provides an analytical foundation to the liquidation strategy assumed in the CFS model.

Proposition 5.4 *Let asset ℓ be liquid ($g_\ell(s_i^\ell, s_{-i}^\ell) = 1$ for all s^ℓ) and asset m be illiquid (g_m be decreasing). Any strategy s_i such that $s_i^\ell < e_i^\ell$ and $s_i^m > 0$ is not optimal for bank i.*

Proof. The Lagrangian of problem (22) is

$$\mathcal{L}_i = \sum_{k=0}^{M} e_i^k g_k(s_i^k, s_{-i}^k) + \sum_{k=0}^{M} \lambda_k(e_i^k - s_i^k) - \mu h_i(s_i, s_{-i}), \quad \mu \in \mathbb{R}.$$

If $s_i^k > 0$, the first order condition related to s_i^k reads

$$\frac{\partial \mathcal{L}_i}{\partial s_i^k} = 0 \Leftrightarrow e_i^k \frac{\partial g_k}{\partial s_i^k}(s_i, s_{-i}) - \lambda_k - \mu\left(\bar{r}\frac{\partial f_i}{\partial s_i^k}(s_i, s_{-i}) - e_i^k \frac{\partial g_k}{\partial s_i^k}(s_i^k, s_{-i}^k)\right) = 0$$

$$\Leftrightarrow e_i^k \frac{\partial g_k}{\partial s_i^k}(s_i, s_{-i}) - \lambda_k + \mu\left(\left((1-\bar{r})e_i^k + \bar{r}s_i^k\right)\frac{\partial g_k}{\partial s_i^k}(s_i^k, s_{-i}^k) + \bar{r}g_k(s_i^k, s_{-i}^k)\right) = 0. \quad (26)$$

For asset ℓ, equation (26) simply reads $\lambda_\ell = \mu\bar{r}$. Therefore, if $s_i^\ell < e_i^\ell$, one has $\lambda_\ell = 0$ and, as a consequence, $\mu = 0$. On the other hand, if $s_i^m > 0$ equation (26) with $\mu = 0$ implies

$$0 \leq \lambda_m = e_i^m \frac{\partial g_m}{\partial s_i^m}(s_i, s_{-i}) < 0, \quad (27)$$

a contradiction. ∎

Consider now under what circumstances it may be optimal for bank i to sell a positive amount of two different illiquid assets ($s_i^\ell > 0$ and $s_i^m > 0$). It follows from inequalities (27), arguing by contradiction, that $\mu \neq 0$. Then we can solve

equation (26) for μ with $k = \ell, m$. Equating the two expressions obtained, we then have

$$\frac{e_i^\ell \dfrac{\partial g_\ell}{\partial s_i^\ell}(s_i^\ell, s_{-i}^\ell) - \lambda_\ell}{\bar{r}\dfrac{\partial f_i}{\partial s_i^\ell}(s_i, s_{-i}) - e_i^\ell \dfrac{\partial g_\ell}{\partial s_i^\ell}(s_i^\ell, s_{-i}^\ell)} = \frac{e_i^m \dfrac{\partial g_m}{\partial s_i^m}(s_i^m, s_{-i}^m) - \lambda_m}{\bar{r}\dfrac{\partial f_i}{\partial s_i^m}(s_i, s_{-i}) - e_i^m \dfrac{\partial g_m}{\partial s_i^m}(s_i^m, s_{-i}^m)}.$$

As a consequence, we obtain the following necessary (and sufficient, under suitable assumptions; see Remark 5.3) condition for an interior solution in the assets ℓ and m, i.e. a liquidation strategy s_i such that $s_i^\ell \in (0, e_i^\ell)$ and $s_i^m \in (0, e_i^m)$:

$$\frac{e_i^\ell \dfrac{\partial g_\ell}{\partial s_i^\ell}(s_i^\ell, s_{-i}^\ell)}{e_i^m \dfrac{\partial g_m}{\partial s_i^m}(s_i^m, s_{-i}^m)} = \frac{\bar{r} g_\ell(s_i^\ell, s_{-i}^\ell) + ((1-\bar{r})e_i^\ell + \bar{r}s_i^\ell)\dfrac{\partial g_\ell}{\partial s_i^\ell}(s_i^\ell, s_{-i}^\ell)}{\bar{r} g_m(s_i^m, s_{-i}^m) + ((1-\bar{r})e_i^m + \bar{r}s_i^m)\dfrac{\partial g_m}{\partial s_i^m}(s_i^m, s_{-i}^m)}. \tag{28}$$

The previous condition shows that the optimality of a liquidation strategy that is an interior point in (at least) two assets depends on the optimal strategies of other banks and all the parameters of the model: the inverse demand function and its derivative, the initial endowments of bank i and the minimum leverage ratio set by the regulator. On the other hand, the same equality may be used to derive sufficient conditions under which an interior solution may not be optimal. An example is provided in the following remark, under additional assumptions on the inverse demand functions.

Remark 5.5 (a) In the exponential case[25]

$$g_k(s_i^k, s_{-i}^k) = \exp\left(-\alpha_k \sum_{j=1}^N s_j^k\right), \quad \alpha_k > 0, \quad k = \ell, m,$$

used in the CFS model for the single-asset case, condition (28) simplifies to

$$\frac{\alpha_\ell e_i^\ell}{\alpha_m e_i^m} = \frac{\alpha_\ell((1-\bar{r})e_i^\ell + \bar{r}s_i^\ell) - \bar{r}}{\alpha_m((1-\bar{r})e_i^m + \bar{r}s_i^m) - \bar{r}},$$

from which, after a few algebraic manipulations,

$$\frac{\alpha_\ell e_i^\ell}{\alpha_m e_i^m} = \frac{1 - \alpha_\ell s_i^\ell}{1 - \alpha_m s_i^m}. \tag{29}$$

This necessary condition has the remarkable property that it does not depend on the liquidation strategies adopted by other banks (and it does not depend on the regulatory ratio $\bar{r}$ either).

Moreover, it clearly shows that an optimal liquidation strategy should not

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[25]In this remark and Remark 5.6, we are assuming that when no asset sales have occurred, all prices are normalized to one, as in the CFS model.

be based only on the minimum price that an asset k can attain, or on the velocity with which its price deteriorates (as measured by the derivative of the inverse demand function, or its semi-elasticity α). The quantity that matters is instead the product $\alpha_k e_i^k$ of the semi-elasticity times the endowment of asset k initially held by bank i, i.e. a transformation of the maximum price deterioration that bank i could cause by selling the whole of its endowment of asset k.

Finally, notice that equation (29) can be used to derived sufficient conditions under which an interior strategy in assets ℓ and m can not be optimal. Indeed, since

$$1 - \alpha_\ell e_i^\ell \leq \frac{1 - \alpha_\ell s_i^\ell}{1 - \alpha_m s_i^m} \leq \frac{1}{1 - \alpha_m e_i^m}, \quad \forall s_i^\ell \in [0, e_i^\ell], s_i^m \in [0, e_i^m],$$

if

$$\frac{1}{1 - \alpha_m e_i^m} < \frac{\alpha_\ell e_i^\ell}{\alpha_m e_i^m}, \quad \text{or} \quad \frac{\alpha_\ell e_i^\ell}{\alpha_m e_i^m} < 1 - \alpha_\ell e_i^\ell,$$

then equation (29) can not hold.

(b) Equation (28) simplifies also in the case of affine inverse demand functions

$$g_k(s_i^k, s_{-i}^k) = 1 - \alpha_k \left(\sum_{j=1}^{N} s_j^k \right), \quad \alpha_k > 0, \quad k = \ell, m,$$

reading

$$\frac{\alpha_\ell e_i^\ell}{\alpha_m e_i^m} = \frac{1 - \alpha_\ell \left(s_i^\ell + \sum_{j=1}^{N} s_j^\ell \right)}{1 - \alpha_m \left(s_i^m + \sum_{j=1}^{N} s_j^m \right)}.$$

While in this case the strategy of bank i depends on the strategies of other players, comments similar to those made in the exponential case apply.

Since it may be the case that no interior solution is optimal, it is useful to derive optimality conditions that allow to compare boundary solutions as well. This can be done evaluating the objective function of problem (22) directly. Indeed, consider two feasible points for that problem that differ only in the two components ℓ and m, namely s_i and $\hat{s}_i$ such that $s_i^\ell \neq \hat{s}_i^\ell$, $s_i^m \neq \hat{s}_i^m$ and $s_i^k = \hat{s}_i^k$ for all $k \neq \ell, m$. By direct evaluation of the objective function, s_i is a better strategy than $\hat{s}_i$ if and only if

$$\sum_{k=0}^{M} e_i^k g_k(s_i^k, s_{-i}^k) > \sum_{k=0}^{M} e_i^k g_k(\hat{s}_i^k, s_{-i}^k) \Leftrightarrow \sum_{k=0}^{M} e_i^k \big(g_k(s_i^k, s_{-i}^k) - g_k(\hat{s}_i^k, s_{-i}^k) \big) > 0$$

$$\Leftrightarrow e_i^\ell \big(g_\ell(s_i^\ell, s_{-i}^\ell) - g_\ell(\hat{s}_i^\ell, s_{-i}^\ell) \big) + e_i^m \big(g_m(s_i^m, s_{-i}^m) - g_m(\hat{s}_i^m, s_{-i}^m) \big) > 0. \qquad (30)$$

In case of two boundary solutions with $s_i^\ell > 0$, $s_i^m = 0$, $\hat{s}_i^\ell = 0$ and $\hat{s}_i^m > 0$, it is easy to pick the better one based on condition (30), that now reads

$$\frac{g_m(0, s_{-i}^m) - g_m(\hat{s}_i^m, s_{-i}^m)}{g_\ell(0, s_{-i}^\ell) - g_\ell(s_i^\ell, s_{-i}^\ell)} > \frac{e_i^\ell}{e_i^m}. \qquad (31)$$

Remark 5.6 In the case of affine inverse demand functions, condition (31) simplifies to

$$\frac{\alpha_m e_i^m}{\alpha_\ell e_i^\ell} > \frac{s_i^\ell}{s_i^m}.$$

This condition goes in the same direction as Remark 5.5, confirming the role played by the maximum potential impact of bank i's sales of each illiquid asset when discriminating among different strategies.

5.2 Numerical results

We finally present the results of two simulation exercises in order to illustrate the main points discussed in Section 5.1 on the optimality of different liquidation strategies.

In the first example, we consider a totally interconnected banking system of 10 homogeneous banks. The liability side of their balance-sheet is as in Table 1. On the asset side, each bank is endowed with 120 units of asset 1 and with 50 units of asset 2 (interbank loans amount to 30). We then perform contagion analysis assuming an initial shock that cancels an equal share S of bank 1's endowments of both assets. Keeping the minimum price of asset 2 constant, we let the minimum price of asset 1 and the share S vary. The results are presented in Figure 7.

Focusing on the shares of each asset liquidated by banks that are not hit by the initial shock, we can distinguish three main strategies, depending on the values of the two parameters:

(i) both shares equal zero, meaning that contagion effects are not strong enough to trigger asset sales by other banks – this happens when the initial shock is small in magnitude;

(ii) both shares are equal to one, i.e. contagion effects are so strong that force all banks to completely deleverage and, eventually, default – this case covers the region in which the shock is higher and asset 1 is less liquid;

(iii) both shares are in the interval $(0, 1)$, i.e. an interior point strategy is optimal – this is the case only in two small regions of the parameter space that we are showing, corresponding either to an intermediate value of the initial shock, or a higher degree of liquidity of asset 1. Notice that, in the first case, while the ratio of the quantity of asset liquidated to the initial endowment decreases slowly for asset 1 as its liquidity improves, the decrease is exponential for asset 2. This suggests that, as liquidity conditions of asset 1 improve, its preeminence in optimal liquidity strategies increases (at a more than linear pace).

These outcomes are in line with the claim of Section 5.1 that an optimal liquidation strategy can not be chosen in advance, as it depends crucially on the characteristics of the initial shock and of banks' portfolios. That is, from the point of view of a given bank, liquidity as measured by price elasticity or by the

Figure 7: Contagion analysis with two illiquid assets (1)

(a1) *Share of asset* 1 *sold by bank i*   (a2) *Share of asset* 2 *sold by bank i*
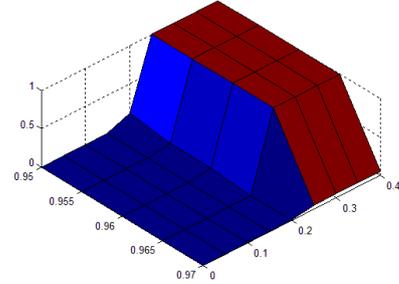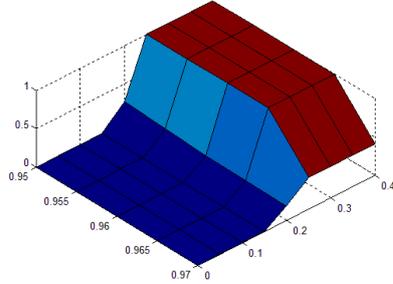
(b1) *Final price of asset* 1   (b2) *Final price of asset* 2
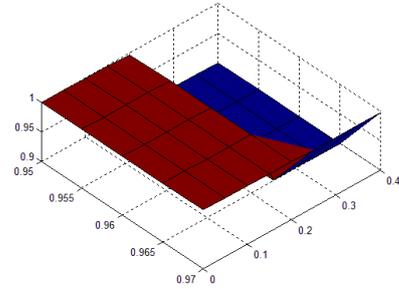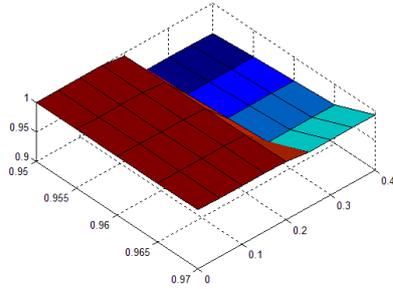
(c1) *Final aggregate asset value*   (c2) *Final loss on non-interbank liabilities*
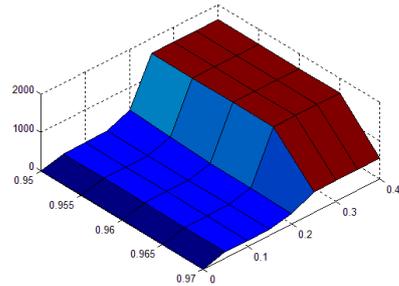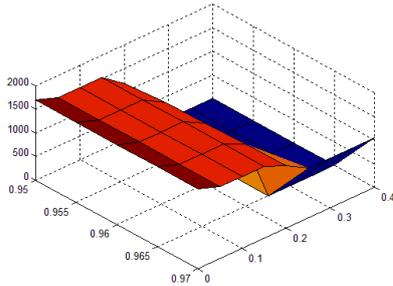
(1) On the x axis the share S of assets of bank 1 canceled by the initial shock is reported. On the y axis, the minimum price that asset 1 can attain. On the z axis, different metrics are plotted: the share of asset 1 and asset 2 sold by banks that are not hit by the initial shock, in panels (a1) and (a2); the price of the assets after contagion has taken place, in panels (b1) and (b2); the final value of total assets, panel (c1); the amount of senior debt that banks are not able to pay back, panel (c2). The main parameters of the model are set as follows: $\bar{r} = 4\%$; $p_{min}(2) = 0.9$.

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minimum attainable price may not be the only relevant parameter on the basis of which a liquidation strategy is planned (unless one perfectly liquid asset is available).

In this connection, another finding that is worth pointing out is that, in the same region of parameter values that yield interior point strategies, the final price of the first asset is lower than the price of the second one. Although in principle asset 1 is more liquid than asset 2 (as indicated by the minimum price that they could attain if all banks completely liquidated their endowments), it may end up with a greater price deterioration due to fire sales. At least in some cases, this might be a socially undesirable outcome, as prices could then provide incorrect signals concerning the intrinsic value of assets.

Another example of how all relevant parameters interact in the choice of the optimal liquidation strategy may be provided in a slightly more complicated setting, in which there are three illiquid assets and each bank holds two of them only in its portfolio. In particular, asset 1 (more liquid) represents a share L of each bank's total assets, while only banks 1 to 5 are endowed with asset 2 (intermediate level of liquidity) and banks 6 to 10 hold asset 3 (less liquid) each. Asset 2 and 3 account for a share M of total assets of the banks holding them. Given an exogenous shock to the holdings of asset 2 in bank 1's portfolio, we want to analyze how the liquidation strategies of other banks are affected as L and M change, while keeping the sum $L + M$ constant (as a consequence, the amount of interbank loans of each bank is constant too).[26]

The results of Table 2 show that the asset hit by the initial shock to bank 1 is never sold by other banks (banks i in the table), unless their selling of asset 1 is not enough to restore the minimum leverage ratio required by the regulator. This is an instance of sequential selling, justified by the significant deterioration of asset 2's price in the aftermath of the initial shock. On the other hand, the behavior of banks 6 to 10 (banks j in the table) is more complex. Unless they default or do not have to deleverage, their optimal strategy is always at some interior point. In particular, the share of assets 1 and 3 that they have to sell is increasing in M (i.e., the ratio of asset 3 – more illiquid – to total assets). However, they grow at a different pace, with the share of asset 1 sold that increases considerably faster as bank's portfolio becomes less liquid.

6   Conclusions

In this paper we build on the model by Cifuentes, Ferrucci, Shin (2005), a standard reference for contagion analysis in banking networks with (market) liquidity effects due to asset fire sales, in order to study how different network topologies respond to exogenous shocks (either systemic or idiosyncratic). First, we provide detailed analytical foundations to the key insights of that model, proving under fairly general assumptions the existence of an equilibrium (in principle there might be several equilibria, but only one of them can actually occur in the

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[26]Balance-sheet dimensions and network topology are the same as in the previous exercise.

Table 2: Optimal liquidation strategies (1)

L	M	Bank i, asset 1	Bank i, asset 2	Bank j, asset 1	Bank j, asset 3
45.0%	40.0%	0.0%	0.0%	0.0%	0.0%
40.0%	45.0%	5.5%	0.0%	0.0%	0.0%
37.5%	47.5%	12.6%	0.0%	0.0%	2.4%
35.0%	50.0%	22.9%	0.0%	5.1%	5.5%
32.5%	52.5%	41.8%	0.0%	17.8%	9.0%
30.0%	55.0%	100%	100%	100%	100%

(1) The first two columns report values of the shares L and M of a bank's assets that are of type 1 and 2 (or 3), respectively. On the other columns, we report optimal liquidation strategies of representative banks from the two groups: bank i stands for banks 2 to 5; bank j stands for banks 6 to 10. The parameters are set as follows: $\bar{r} = 4\%$; $p_{min}(1) = 0.95$; $p_{min}(2) = 0.92$; $p_{min}(3) = 0.90$.

model) and the convergence of the algorithm to compute it. Then, to answer our research question (to what extent the resilience of an interbank network depends on its topology), we run a set of simulation exercises testing how some stylized networks are affected by different types of shocks. Finally, we extend the original model to include the case where banks hold in their portfolios multiple illiquid assets, with different degrees of illiquidity.

The main conclusions and policy implications can be summarized as follows. First, consistent with the relevant literature, our results confirm that the resilience of the banking system may depend heavily on the network topology. From a policy perspective, this might justify that macroprudential authorities have access to granular data on interbank exposures.[27] Second, we show that different network configurations may be affected by changes in relevant market and regulatory parameters (assets' liquidity and minimum leverage ratio, respectively) in different ways. The search for an optimal balance between different policy instruments that can be used to cope with systemic risk would benefit from a deeper knowledge of the sensitivity of the banking network at hand on the key parameters. Finally, in a realistic setting bank assets show heterogeneous degrees of illiquidity (as measured by price elasticity). In this context, there exist several channels of contagion (or, more precisely, interdependence) and banks that take losses and are forced to partially deleverage have to choose what liquidation strategy to adopt. To this end, we show that optimal liquidation strategies (from an equity maximization perspective) may be in disagreement with common behavioral strategies. For a bank that has to deleverage, the liquidity of an asset is not an absolute quality of the asset, but relates to both the composition of the bank's portfolio and the liquidation strategies adopted by other banks.

Future research extending this paper could extend the analysis of the multiple asset setting and include the additional channel of contagion represented by funding liquidity.

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[27]For policy purposes, it is also worth mentioning that the impact of an exogenous shock of given magnitude depends on how it is distributed among the banks in the system (even in the case of homogeneous banks) and the structure of this dependence varies for different network configurations. This has implications, for instance, for the optimal allocation of limited resources for banks' recapitalizations.

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Appendix

In Section 3 we use extensively Tarski's fixed point theorem, that we recall here for convenience of the reader (see e.g. Carter, 2001).

Theorem A.1 (Tarski's fixed point theorem, 1955) *Let X be a non-empty complete lattice. If $f : X \longrightarrow X$ is non-decreasing, then the set of fixed points of f is a non-empty complete lattice. In particular, f has a greatest fixed point $\overline{x}$ and a least fixed point $\underline{x}$.*

We also prove two useful results that we need in Section 3.

Lemma A.2 *Let X be a compact metric space and a complete lattice, and let $f : X \longrightarrow X$ be non-decreasing and continuous. Denote by $\overline{m}$ the least upper bound of X and by $\overline{x}$ the greatest fixed point of f. Then*

$$f^{(n)}(\overline{m}) \xrightarrow{n} \overline{x}.$$

Proof. By definition of $\overline{m}$ and $\overline{x}$, as f is non-decreasing,

$$\overline{x} = f(\overline{x}) \leq f(\overline{m}).$$

By iterated application of f (which is non-decreasing) to the first and last term of the previous chain of inequalities,

$$\overline{x} \leq f^{(n)}(\overline{m}). \tag{A.1}$$

Since X is compact, we can assume without loss of generality that the sequence $(f^{(n)}(\overline{m}))_n$ converges to a limit x^*. By continuity of f,

$$x^* = \lim_n f^{(n)}(\overline{m}) = f\left(\lim_n f^{(n-1)}(\overline{m})\right) = f(x^*),$$

i.e. x^* is a fixed point of f. Taking limits on both sides of (A.1), we obtain $\overline{x} \leq x^*$, from which $\overline{x} = x^*$, by definition of $\overline{x}$. ∎

Lemma A.3 *Let X be a complete lattice and $f, g : X \longrightarrow X$. If f is non-decreasing and $f \leq g$ on X, then*

$$f^{(n)}(x) \leq g^{(n)}(x), \quad \forall x \in X, n \in \mathbb{N} \backslash \{0\}.$$

Proof. We prove the result by induction. If $n = 1$, then $f^{(n)}(x) \leq g^{(n)}(x)$ by hypothesis. Assume that the previous inequality holds for an arbitrary $n \geq 1$. Then, for all $x \in X$,

$$f^{(n+1)}(x) = f\left(f^{(n)}(x)\right) \leq f\left(g^{(n)}(x)\right) \leq g\left(g^{(n)}(x)\right) = g^{(n+1)}(x),$$

where the two inequalities follow from the monotonicity of f and the hypothesis that $f \leq g$, respectively. ∎