Three to Tango

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Abstract

We examine a model with three assets - money, debt and equity- that may be used for transaction purposes. Money serves as the payment instrument, the liquidity of debt and equity is derived as an equilibrium outcome. Depending on the equilibrium liquidity of the assets, open market operations may be effective or ineffective to foster output. When effective, sometimes a monetary expansion, sometimes a contraction is called for, depending on the scarcity of liquidity.

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JEL: E40

1 Introduction

This paper examines the role of liquid private and public instruments in a monetary economy. Three payment instruments are available, namely money, public debt and equity, in an economy with enforcement problems due to lack of commitment and anonymity. The degree of liquidity of these instruments is an endogenous outcome of the model, in the sense that there are different equilibria in which liquidity is either traded freely or trapped. Moreover, even at the equilibrium in which liquidity flows freely among the traders, there are situations in which there is the full spectrum of liquid assets with different returns, from the lowest represented by money to the highest represented by equity with debt in between. Whether the economy is in a

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liquidity trap or not is the result of the coordination of the traders' expectations, but whether there is a full spectrum of interest rates depends on the overall availability of liquidity.

The events of the last fifteen years have shown that the assets liquidity may have major consequences for the business cycle.[1] Liquidity has been examined from different theoretical perspectives. For liquidity to play any role in the allocation of resources, the Arrow-Debreu benchmark has to be abandoned. Early contributions, such as Diamond and Dybvig (1983) and Holmstrom and Tirole (1998), have considered models in which the traders face liquidity shocks either for consumption or production purposes, absent well functioning markets for insurance due to traders' private information about the shocks. In these circumstances, private bank liabilities or public debt may play a role in the allocation of resources. Money, instead, has no special role to play in such environments.[2] Other papers have taken the perspective that liquidity serves to self-insure against shocks, in a Bewley model.[3] In Kiyotaki and Moore (2019), money and equity are available for trade but equity is not fully liquid for exogenous reasons. In such a world, money may be used to compensate for the illiquidity of equity. Our paper takes the opposite perspective, placing money center stage as the payment instruments and deriving an endogenous liquidity role for debt and equity.[4] In particular, our paper exploits the fact that in monetary economies with uncertain trading prospects money holdings may be misallocated when most useful. In this line, the closest papers are in the monetary literature. In Kocherlakota (2003), illiquid public debt serves to reshuffle misallocated money.[5] The novelty of our paper is that the liquidity of the assets is an equilibrium phenomenon rather

[1] For some evidence on the connection between public and private liquidity and the 2008 financial crisis, see Dick-Nielsen, Feldhutter and Lando (2012). On the connection between liquidity and business cycles, see Naes, Skjeltorp and Odegaard (2011).

[2] Williamson (2012) is an attempt to study money and debt in an environment that shares some features with Diamond and Dybvig (1983). For the limitations of such an attempt, see Zannini (2020).

[3] Bewley (1980) considers a monetary economy. Townsend (1980) presents a deterministic version of the Bewley model. Woodford (1990) considers a similar economy with debt and capital but no money. Townsend (1987) with money and capital but no debt. In the model of Brunnermeier and Sannikov (2016a) there is only capital with credit constraints as in Kiyotaki and Moore (1997).

[4] Cui and Radde (2020) have endogenized the notion of liquidity of money and equity of Kiyotaki and Moore (2019) using a search model. Our model has three assets and a different notion of endogenous liquidity as an equilibrium dependent feature.

[5] Several papers have exploited this feature in the monetary literature, including Berentsen, Camera and Waller (2007), Ferraris and Watanabe (2008), Lagos and Rocheteau (2008), Geromichalos and Herrenbrueck (2016, 2017), and Araujo and Ferraris (2020).

than an exogenous assumption and there is a third asset, capital, that may be used as a liquid instrument as well. The presence of a liquidity premium on capital boosts capital accumulation, output and welfare relative to a world in which only money and debt are liquid. This is because the use of equity as a liquid instruments induces a pecuniary externality, lowering the price of second hand capital, with beneficial effects on output.

The model gives rise to two equilibria. In the first, money debt and equity are perfect substitutes as direct payment instruments to acquire second hand capital. The interest rate on debt is nil, capital accumulation and output are inefficiently low, monetary policy is ineffective. Accordingly, we interpret this equilibrium as a liquidity trap.[6] In the second equilibrium, debt helps reshuffle misallocated money which is used as a direct payment instrument for capital in the secondary market. Equity may serve either as a substitute of debt and complement of money or viceversa, depending whether the overall liquidity is scarce or plentiful. At this equilibrium, the interest rate on debt is positive, capital and output are larger than in the previous equilibrium and monetary policy is effective. In this equilibrium liquidity flows freely among traders, but liquidity may be scarce. Which equilibrium emerges depends on the coordination of the traders expectations, but once they coordinate on the liquid equilibrium, whether liquidity is scarce or not depends on policy.

Recently, Brunnermeier and Sannikov (2016b) found that money creation induces a Tobin effect on capital,[7] with positive consequences for welfare, in a Bewley economy with financial frictions. In our economy, open market operations can be examined since both money and public debt are available.[8] Expansionary open market operations have a liquidity effect as in Lucas (1990) and are unambiguously beneficial in terms of capital, output and welfare when the economy is not liquidity constrained, but not necessarily when the economy is liquidity constrained.[9] When the economy is in a liquidity trap, only fiscal policy is effective.

Section 2 presents the model, Section 3 the efficient benchmark and Section 4 the monetary equilibria. Section 5 compares the returns of the assets. Section 6 discusses policy. Section 7 concludes. The proofs are in the Appendix.

[6]For a definition of a liquidity trap, see Eggertsson (2008). For an alternative approach see Andolfatto and Williamson (2015).

[7]See Tobin (1969) for the original argument.

[8]See Rocheteau, Wright and Xiao (2018) for a monetary model with open market operations.

[9]See Herrenbrueck (2019) for another paper on the liquidity channel of monetary policy.

2 Model

2.1 Fundamentals

Time is discrete and continues for ever. The economy is inhabited by a unit mass of anonymous ex-ante identical agents, who cannot commit to future actions. Two different goods are available in the economy. One good is durable and can be either consumed or accumulated as capital, k. There is an initial amount of such a good evenly distributed among the agents. All the agents can produce the durable good with labor used as input into a linear production function. Labor generates linear disutility for the agents. All the agents wish to consume the durable good from whose consumption they derive linear utility. The other good is perishable and its production requires capital as an input. The perishable good is produced with a differentiable production function $f(k)$ that satisfies $f(0) = 0$, $f'(0) = \infty$ and $f'(\infty) = 0$, with $f' > 0 > f''$. At the beginning of each period, half of the agents are randomly selected by an i.i.d. process to become producers of the perishable good. Capital depreciates after production at rate $\delta < 1$. If not used in production, capital can be stored without cost. Only the producers of the perishable good wish to consume it, obtaining linear utility from its consumption. The perishable good disappears if not consumed immediately after production. We call these agents entrepreneurs and the rest investors. All the agents discount future payoffs at rate $\beta < 1$.

2.2 Trade

In each period the agents can interact in three different markets, a liquidity market (LM), a secondary market (SM), and a primary market (PM), opening sequentially. In the LM, the agents may choose to trade liquidity at a competitive price p. There is neither production nor consumption at this stage. Capital accumulated from the past can be traded in the SM market at a competitive price q. Production and consumption of the perishable good occur before the PM opens. The durable good is the numeraire and can be traded in the competitive PM. Production and consumption of the durable good occur before the beginning of the following period. Three assets are available for trade, money, m, public bonds, b, and capital, k. Money is an intrinsically worthless durable object whose supply, M, is controlled by the the monetary authorities. Its price in numeraire units is v. Bonds, whose supply is denoted by B, are sold by the fiscal authority for money during the PM of each period at a competitive price ρ and reimbursed in money during the following PM. Capital is accumulated at the end of the PM. Its aggregate stock is denoted with K. Capital is identifiable and can be seized by outsiders, hence, the agents can issue equity on its undepreciated value. Equity issued in a given period disappears before the beginning of the next period.

2.3 Government

The monetary and fiscal authority are consolidated in a single agency called the government. The government has a limited ability to tax the agents due to their anonymity. However, since capital is identifiable by outsiders, the government may be able to tax the agents up to the value of their undepreciated capital holdings. Lump-sum taxation is represented by τ as a fraction of the outstanding stock of liquid government instruments, with a negative τ representing a subsidy. The government fulfills its budget constraint, $M_{+1} = M + B - \rho B_{+1} - \tau (M + B)$. We define the bonds to money ratio $x \equiv \frac{B}{M}$ and assume that the government keeps it constant over time. With this notation, the budget constraint of the government can be written as

$$\frac{M_{+1}}{M} = \frac{(1+x)(1-\tau)}{1+\rho x}. \tag{1}$$

Hence, we will use x as the monetary policy parameter, capturing open market operations, namely relative shift in the long run proportion of the stock of money and bonds available in the economy, and indirectly controlling the growth rate of the money stock. The fiscal policy parameter τ also partially controls the growth rate of the money stock. The policy pair (τ, x) is decided and implemented once and forever at the initial date.

3 Efficiency

Every period, a randomly selected half of the agents turns out to have an amount of capital for which they have no immediate use, while the rest can use it in production. The efficient level of capital accumulation solves the recursive problem,

$$V(k) = \max \ \frac{1}{2}\left[f(2k) + 2k(1-\delta)\right] - k_{+1} + \beta V(k_{+1}), \tag{2}$$

where $2k = K$. Optimization of (2) with respect to capital accumulation leads to the Euler condition

$$1 = \beta \left[f' \left(K_{+1} \right) + 1 - \delta \right]. \tag{3}$$

By continuity and the Inada conditions, a positive solution of (3) exists and by concavity of the production function is unique. The efficient amount of capital is derived from (3) as

$$K^* = f'^{-1} \left(\frac{1 - \beta + \beta\delta}{\beta} \right). \tag{4}$$

The efficient capital accumulation, (4), could be achieved as an equilibrium of a decentralized economy with full commitment always or without anonymity for β sufficiently large.

4 Monetary Equilibrium

Since every period half of the agents have an amount of capital for which they have no immediate use, while the rest can use it in production, there is a motive to trade capital in a secondary market before production of the perishable good occurs. However, trade in such a market is impeded by the inability of the agents to commit themselves to future actions and by anonymity. There are two imperfections. First, the output $f(k)$ generated with capital is perishable and specific for the use of the entrepreneurs, hence, it cannot be pledged to the investors to acquire k in the secondary market. Second, the agents cannot credibly commit to pay for transactions that occur in the LM or SM with work done in the PM or in future periods. This is because the agents are unable to commit to deliver on promises of future payments, and anonymous, hence, neither bilateral nor multilateral credit deals can be enforced threatening to punish defectors. In this situation, physical assets play a useful role as payment instruments. There are two main ways in which trade in the secondary market for capital can occur. The agents may choose to use money, debt and equity indifferently as payment instruments to trade capital or they may decide to reallocate money using debt and equity before trading capital for money. We consider each scenario in turn. We identify the first scenario with a situation known in the literature as a liquidity trap and the second scenario with a situation in which liquidity flows normally from investors to entrepreneurs.

4.1 The Trap

First, consider the scenario in which the agents use indifferently money and debt to trade capital in the secondary market. In this case, the agents skip trade in the LM and show up directly for trade in the SM. A competitive equilibrium requires the agents to optimize taking prices as given and the prices to clear markets. We consider stationary equilibria, namely equilibria in which real variables are time invariant. An entrepreneur chooses how much capital to buy in the secondary market k^d to solve

$$U^E(m,b,k) = \max\ f\left(k+k^d\right) + (1-\delta)\left(k+k^d\right) - qk^d + W(m,b)$$

where $W(m,b)$ is the value brought into the primary market, subject to the constraint

$$qk^d \leq vm + vb + k(1-\delta), \tag{5}$$

where the value of the capital acquired in the secondary market cannot exceed the total value of the liquid assets held by the agent, namely money, debt and equity issued on the undepreciated capital stock. An investor sells capital in the secondary market, k^s, to maximize

$$U^I(m,b,k) = \max\ k - k^s + qk^s + W(m,b)$$

subject to the constraint

$$k^s \leq k, \tag{6}$$

since the investor can sell at most the capital held at that point in time. In the primary market, an agent chooses money, m_{+1}, bonds, b_{+1} and capital, k_{+1}, to solve

$$W(m,b) = \max\ vm + vb - t - vm_{+1} - v\rho b_{+1} - k_{+1} + \beta V(m_{+1}, b_{+1}, k_{+1}), \tag{7}$$

where $t \equiv \tau v(M+B)$ and the value function satisfies

$$V(m,b,k) = \max\ \frac{1}{2}\left[U^E(m,b,k) + U^I(m,b,k)\right] + W(m,b). \tag{8}$$

The distribution of assets is degenerate at equilibrium by virtue of the linearity of the payoffs.[10] Since money and debt are perfect substitutes, by arbitrage, the price

[10] As in Lagos and Wright (2005).

of the bond is

$$\rho = 1. \tag{9}$$

The rest of the equilibrium system consists of two equations, namely the Euler condition for money holdings

$$1 = \frac{\beta}{2} \frac{1}{1-\tau} \left[\frac{f'(K) + 1 - \delta}{q} + 1 \right], \tag{10}$$

since money is used by the entrepreneurs to acquire capital in the secondary market and is brought by the investors to the primary market, where the return of money is $\frac{1}{1-\tau}$ by (1) with $\rho = 1$; and the Euler condition for capital accumulation

$$1 = \frac{\beta}{2} \left\{ f'(K) + 1 - \delta + \left[\frac{f'(K) + 1 - \delta}{q} - 1 \right] (1 - \delta) + q \right\}, \tag{11}$$

since capital is used in production and as a liquid instrument in the secondary market by the entrepreneurs and sold by the investors in the secondary market at price $q \geq 1$. Since capital depreciates only in production, the price of capital in the secondary market cannot fall below 1, otherwise no investor would want to sell it. Next, we define a stationary equilibrium with money and bonds used as substitute instruments of trade.

Definition 1 *A stationary liquidity trap equilibrium is a time invariant triple* (ρ, q, K) *such that (9), (10) and (11) hold.*

The following Proposition proves the existence and uniqueness of such an equilibrium.

Proposition 1 *For* $\tau > 0$*, there exists a value* $\widehat{\delta} < 1$*, such that, if* $\delta \in \left[\widehat{\delta}, 1\right)$*, a stationary liquidity trap equilibrium exists and is unique.*

Next, we derive the equilibrium prices and capital accumulation. Define $z \equiv 1 - \tau - \beta \geq 0$. The only stationary equilibrium with trade in the secondary market has $\widehat{\rho} = 1$, namely zero interest on bonds, by (9); by (10) and (11) together,

$$\widehat{q} = \frac{1 - z(1-\delta)}{z + \beta}, \tag{12}$$

namely the Tobin's q; and capital accumulation is given by

$$\widehat{K} = f'^{-1}\left(\frac{2\left(1+\beta-\beta\delta\right)\left(z+\beta\right)-2\left(1-\delta\right)\left(z+\beta\right)^{2}-\beta\left(1+\beta-\beta\delta\right)}{\beta\left(z+\beta\right)}\right). \quad (13)$$

The next Proposition shows that the allocation is inefficient when liquid capital is relatively scarce, due to depreciation.

Proposition 2 *At the liquidity trap equilibrium, if $z > 0$, $\widehat{K} < K^{*}$. For $z = 0$, $\widehat{K} = K^{*}$. There exists a value $\overline{\delta} < 1$, such that, for $\delta > \overline{\delta}$, $z = 0$ cannot be achieved.*

Hence, when depreciation is not too small, taxation is limited and the efficient allocation cannot be achieved. This is an equilibrium in which the liquidity market does not operate, money, debt and equity are perfect substitutes as payment instruments to acquire capital in the secondary market, the nominal interest rate is nil and the allocation is distorted relative to efficiency. Taxation is limited by the undepreciated stock of capital, giving rise to the constraint on taxation. The case $z = 0$, i.e. $1 - \tau = \beta$, corresponds to the so-called Friedman rule, which may not be feasible in this environment due to its informational imperfections. Notice that there is another arrangement, in which equity issued on the undepreciated capital stock is used in the LM to trade money and, then, public debt and money are traded indifferently in the SM for capital. The equilibrium allocation of capital is identical to (13), but the binding constraints imply that this arrangement can be sustained only as a knife edge case.

4.2 The Flow

Next, we examine the case in which money and the other liquid assets are traded in the liquidity market before capital is exchanged in the secondary market. In other words, liquidity flows toward the traders who most need it. There are two situations to be considered in this case. In the first scenario, public debt and equity are used indifferently to trade money in the LM. This situation arises when the available stock of debt is scarce relative to the stock of money, so that the traders use equity to try relax the liquidity constraint. In the second scenario, public debt is used to trade money in the LM and money and equity are used indifferently to trade capital in the SM. This situation arises when the available stock of debt is plentiful relative to the stock of money. We call these two situations, the scarce and plentiful liquidity cases, respectively.

4.2.1 Scarce Liquidity

The next trading arrangement works as follows. First, the agents visit the LM. The entrepreneurs acquire money paying with debt and equity issued on the undepreciated PM value of their current capital holdings.[11] Then, with the money holdings accumulated from the previous period and the extra money just acquired, the entrepreneur buys capital in the secondary market. A competitive equilibrium requires the agents to optimize taking prices as given and the prices to clear markets. Again, we consider stationary equilibria. An entrepreneur chooses how much money to acquire in the liquidity market, m^d, and how much capital to buy in the secondary market, k^d, to maximize

$$U^E(m,b,k) = \max\ f\left(k+k^d\right) + (1-\delta)\left(k+k^d\right) - qk^d + vm^d - pm^d + W(m,b)$$

subject to the constraints

$$pm^d \leq vb + (1-\delta)k. \tag{14}$$

in the LM and the constraint

$$qk^d \leq vm + vm^d, \tag{15}$$

in the SM. These two constraints reflect the purchase of money in the liquidity market using debt and equity on undepreciated capital and, then, the purchase of new capital in the secondary market using all the money held by the entrepreneur, including the cash just acquired in the liquidity market. An investor sells money in the liquidity market, m^s, and capital in the secondary market, k^s, to maximize

$$U^I(m,b,k) = \max\ k - k^s + qk^s + pm^s - vm^s + W(m,b)$$

subject to the constraint

$$vm^s \leq vm, \tag{16}$$

[11] For simplicity, we assume that equity is one period lived and disappears at the end of the PM. Alternatively, one could think of one epriod private debt.

and

$$k^s \leq k. \tag{17}$$

These constraints reflect the limited amount of cash and capital currently in the hands of the investor. The intra-period value function W is given by (7), and the beginning of period value V by (8). In the Appendix, we show that this case arises when $x < 1$ and the depreciation of capital is not too small, i.e. precisely when liquidity is scarce. This, in turn, implies that (14) is binding. Since money and bonds are traded for each other in the liquidity market, by arbitrage, their returns are equated, i.e. $\frac{p}{v} = \rho^{-1}$. The other stationary equilibrium conditions for this case are the Euler condition for money

$$1 = \frac{\beta}{2} \frac{1 + \rho x}{(1 - \tau)(1 + x)} \left[\frac{f'(K) + 1 - \delta}{q} + \rho^{-1} \right], \tag{18}$$

since money is used by the entrepreneurs to acquire capital in the secondary market and sold for bonds by the investors in the liquidity market earning $\frac{p}{v} = \rho^{-1}$; the Euler condition for capital accumulation

$$1 = \frac{\beta}{2} \left\{ f'(K) + 1 - \delta + \left[\frac{f'(K) + 1 - \delta}{q\rho^{-1}} - 1 \right] (1 - \delta) + q \right\}, \tag{19}$$

since capital is used as input in production and as liquid instrument in the LM by the entrepreneurs and sold by the investors in the secondary market at price q; and the binding liquidity constraint (14)

$$qx + 2(1 - \delta) = q\rho^{-1}. \tag{20}$$

Notice the difference with the liquidity trap case. First, in the Euler conditions for liquidity, (18), the RHS contains the term ρ^{-1}, which reflects the fact that the investors can sell their money holdings that would otherwise remain idle for bonds earning a payoff from them if the price is smaller than 1, i.e. there is an interest rate. On the other hand, the return on money may be smaller than in the previous case precisely when the interest on the bond is positive. Second, in the Euler condition for capital accumulation, (19), the liquidity premium for capital includes the interest rate of the bond. Next, we define the equilibrium with scarce liquidity.

Definition 2 *A stationary equilibrium with scarce liquidity is a time invariant triple* (ρ, q, K) *such that (18), (19) and (20) hold.*

The next Proposition establishes existence and uniqueness of such an equilibrium with positive interest.

Proposition 3 *There exist values* $\underline{\delta} < 1$ *and* $\underline{x} < 1$*, such that, if* $\delta \in [\underline{\delta}, 1)$ *and* $x \in [\underline{x}, 1)$*, a stationary equilibrium with scarce liquidity exists and is unique.*

This is an equilibrium in which the liquidity market is open, debt and equity are substitutes as instruments to trade money in such a market, while money is the only payment instrument to acquire capital in the secondary market, the nominal interest rate is positive. The capital accumulation is still inefficiently small but higher than in the liquidity trap case.

4.2.2 Plentiful Liquidity

In the next scenario, the bonds, which are relatively plentiful, are used to trade money in the LM and, then, money and equity are traded in the SM for capital. This scenario leads to the following constraints for an entrepreneur. First, the constraint

$$pm^d \leq vb. \tag{21}$$

in the LM and the constraint

$$qk^d \leq vm + vm^d + (1-\delta)\,k, \tag{22}$$

in the SM. These two constraints reflect the purchase of money in the liquidity market using debt and, then, the purchase of new capital in the secondary market using all the money held by the entrepreneur, including the cash just acquired in the liquidity market and equity on undepreciated capital. The rest of the equations remain as in the scarce liquidity case. Since money and bonds are traded for each other in the liquidity market, by arbitrage, their returns are equated, i.e. $\frac{p}{v} = \rho^{-1}$. The other stationary equilibrium conditions for this case are the Euler condition for money

$$1 = \frac{\beta}{2}\frac{1+\rho x}{(1-\tau)(1+x)}\left[\frac{f'(K)+1-\delta}{q} + \rho^{-1}\right], \tag{23}$$

since money is used by the entrepreneurs to acquire capital in the secondary market and sold for bonds by the investors in the liquidity market earning $\frac{p}{v} = \rho^{-1}$; the Euler condition for capital accumulation

$$1 = \frac{\beta}{2}\left\{f'(K) + 1 - \delta + \left[\frac{f'(K) + 1 - \delta}{q} - 1\right](1-\delta) + q\right\}, \tag{24}$$

since capital is used as input in production and as liquid instrument in the LM by the entrepreneurs and sold by the investors in the secondary market at price q; and the complementary slackness condition for the liquidity constraint (21)

$$\left[f'(K) + 1 - \delta - q\rho^{-1}\right]\left(x - \rho^{-1}\right) = 0. \tag{25}$$

Notice the difference with the scarce liquidity case. First, in the Euler condition for capital accumulation, (24), the liquidity premium for capital does not include the interest on debt, since equity is now used as a substitute of money to acquire directly capital. Second, the liquidity constraint (21) may be binding or not. Next, we define the equilibrium with money, bonds and equity as complements. Next, we define the equilibrium with plentiful liquidity.

Definition 3 *A stationary equilibrium with plentiful liquidity is a time invariant triple (ρ, q, K) such that (23), (24) and (25) hold.*

The next Proposition establishes existence and uniqueness of such an equilibrium with positive interest.

Proposition 4 *For $x \geq 1$, a stationary equilibrium with plentiful liquidity exists and is unique.*

There are two possibilities in this case, as the liquidity constraint may be binding or not. There is a cutoff value of x such that the equilibrium is constrained or not depending whether the bonds to money ratio is smaller or larger than the cutoff, as the next Proposition documents.

Proposition 5 *There exists a value $\overline{x} > 1$ such that the stationary equilibrium with plentiful liquidity is constrained if $x \in [1, \overline{x})$ and unconstrained if $x \in [\overline{x}, \infty)$.*

First, consider the case in which the liquidity constraint is binding. At the constrained equilibrium, the price of the bond is given by

$$\widetilde{\rho} = \frac{1}{x}, \tag{26}$$

which is smaller than 1 and, thus, the nominal interest rate on bonds is positive, whenever $x > 1$; Tobin's q is given by

$$\widetilde{q} = \frac{2 - z(1-\delta)(1+x)}{2\beta + z(1+x)}, \tag{27}$$

and capital by

$$\widetilde{K} = f'^{-1}\left(\frac{2\beta(1-\beta+\beta\delta) + z(2-\beta+\beta\delta)(1+x)}{2\beta^2 + z\beta(1+x)}\right). \tag{28}$$

At the unconstrained equilibrium, the price of the bond is given by

$$\widetilde{\rho} = \frac{\beta}{\beta + z(1+x)}, \tag{29}$$

which is smaller than 1, and, thus, the nominal interest rate on bonds is positive; Tobin's q is still given by (27); and capital accumulation is still given by (28). Therefore, the real allocation does not change whether the constraint is binding or not, only the nominal interest rate does. The next Proposition shows when the capital stock is higher than in previous cases.

Proposition 6 *At the equilibrium with plentiful liquidity, there exists a value* $\widetilde{x}$ *such that, for* $x > \widetilde{x}$, $\widehat{K} < \widetilde{K} < K^*$, *if* $z > 0$. *For* $z = 0$, $\widehat{K} = \widetilde{K} = K^*$. *When* $\delta > \overline{\delta}$, $z = 0$ *cannot be achieved.*

Since there is a one-to-one correspondence between capital accumulation, output and welfare, the equilibrium in which liquidity is not trapped is at least weakly superior to the liquidity trap. When taxation is limited by the available capital stock, due to depreciation, the efficient allocation cannot be achieved and the equilibrium with unhindered liquidity is strictly superior to the liquidity trap when liquidity is not too scarce. This is an equilibrium in which the liquidity market is open, debt serves to reshuffle money holdings in such a market, while money and equity are used as substitute payment instruments to trade capital in the secondary market. In this case, there is a positive nominal interest rate and capital accumulation is higher than before. Notice that this last result would not hold if equity were not used as payment instrument alongside money, since the price of the bond would be the same but Tobin's q would be much larger, $q = 2\left[2\beta + z(1+x)\right]^{-1}$. This would be enough to depress trade in the secondary market and, consequently, drive capital accumulation below the level of the liquidity trap equilibrium. In sum, the use of three instruments partly as complements partly as substitutes is essential to boost capital, output and welfare relative to the situation in which all three are substitutes as payment instruments in the secondary market. In turn, capital and output are larger with three instruments relative to a situation in which only money and bonds are used as payment instruments because of the liquidity premium generated for capital. This can be seen directly from (11) and (24). The use of capital as a liquid instrument lowers Tobin's q, through a sort of pecuniary externality, with beneficial effects on capital accumulation and output.

4.3 Self-fulfilling Beliefs

In this model, whether the economy is in a scarce or plentiful liquidity situation depends on the behavior of the public authorities, through the policy parameter x, while being in a situation in which liquidity flows normally throughout the economy or in a liquidity trap depends on how the traders coordinate their expectations. To illustrate how agents may end up coordinating some of the time on the equilibrium in which liquidity flows freely throughout the economy and some other time on the liquidity trap, we introduce sunspot events in the spirit of Cass and Shell (1983), namely payoff irrelevant events that can affect the fundamentals only through agents' expectations. We limit attention to stationary sunspot events of order two. Consider two sunspot states, that alternate randomly over time, with probability $\phi_{ss'}$ that next period the event $s' = 1, 2$ will occur, given that today the event $s = 1, 2$ has occurred. For simplicity, we limit attention to the unconstrained case for the equilibrium with plentiful liquidity and without taxation. Let π_s be the inflation rate in state s implied by (1), with $\pi_1 = 1 + (1 - \beta)\, x$ and $\pi_2 = 1$. To simplify the notation, define $\sigma_1 \equiv [1 + (1 - \beta)\, x]\, \beta^{-1}$, $\sigma_2 \equiv 1$, and $\Delta \equiv [2 - \beta + \beta\delta]\, \beta^{-1}$. Suppose that all the agents believe that, if the first event occurs, liquidity flows among traders, while if the second occurs, the liquidity trap will prevail. In other words, if the first sunspot event occurs, all the agents believe that the price of the bond is smaller than 1, while if the second event occurs the price of the bonds is 1. The equilibrium conditions mix the equilibrium conditions for the liquidity trap with those for the liquidity flow using the probabilities of switching sunspot states as follows

$$q_s f'^{-1}(\Delta - q_s) = \frac{\beta}{2} \sum_{s'=1,2} \frac{\phi_{ss'}}{\pi_{s'}} \left[f'^{-1}(\Delta - q_{s'}) + q_{s'} \sigma_{s'} \right], \tag{30}$$

for both s. Next, we define a two state stationary sunspot equilibrium.

Definition 4 *A stationary sunspot equilibrium of order two is a time invariant pair (q_1, q_2) such that (30) hold for $s = 1, 2$ with $q_1 \neq q_2$.*

The next Proposition shows that such an equilibrium exists.

Proposition 7 *A stationary sunspot equilibrium of order two exists.*

Due to self-fulfilling expectations, the economy may end up oscillating randomly between normal times and the liquidity trap, which may arise periodically from the coordination of the agents' expectations on a dominated equilibrium. The only role the government may play in this situation consists in trying to focus the traders' expectations on the good equilibrium in which liquidity is not trapped.

5 Interest Rates

Next, we show that in this economy there is a spectrum of interest rates, since the three assets have different returns depending on which equilibrium prevails, whether liquidity is scarce or plentiful when the economy is not in a liquidity trap and whether the liquidity is constrained or not. Define $r_m \equiv \frac{v_{+1}}{v}$ as the return of money, which is the inverse of the inflation rate, $r_m = \pi^{-1}$, let $r_b \equiv (\rho\pi)^{-1}$ be the return of bonds and $r_k \equiv [f'(K) + 1 - \delta](q\pi)^{-1}$ be the return of capital. Suppose lump-sum taxation is limited by the available undepreciated capital stock, so that $1 - \tau > \beta$. There are three scenarios to analyze. First, consider the liquidity trap equilibrium. In this case, we have that the returns of money and bonds are equal to each other but smaller than the return of capital

$$r_m = r_b < \beta^{-1} < r_k.$$

This is because money, debt and equity are perfect substitutes, and serve to acquire capital in the secondary market. The return of capital has both a fundamental and a liquidity component. Next, consider the liquid equilibrium. In this case, there are two possibilities, with scarce or plentiful liquidity, respectively. If liquidity is scarce, i.e. $x < 1$, the real return on capital is larger than the return of the bond, which, in turn, is larger than the return of money, giving

$$r_m < \beta^{-1} < r_b < r_k.$$

This is because money and bonds are complements and there is a liquidity premium for capital over debt since liquidity is relatively scarce. This ordering of the returns remains the same when liquidity is moderate, i.e. when $1 \leq x < \overline{x}$. When liquidity is abundant, i.e. when $x \geq \overline{x}$, then, the return of bonds and capital are equated and we have

$$r_m < \beta^{-1} = r_b = r_k.$$

The liquidity premium of equity with respect to debt vanishes in this case as the liquidity constraint is not binding anymore. Should $1 - \tau = \beta$ be feasible and implemented, all the returns would be equated

$$r_m = r_b = r_k = \beta^{-1},$$

and the efficient allocation of capital would be achieved. In sum, we have that the real interest rates may all differ from each other or be partially equated depending on the equilibrium in which the economy operates. In this sense, the spectrum of interest rates of liquid instruments is endogenized in this model.

6 Optimal Policy

Next, we discuss optimal fiscal and monetary policy. Due to the linearity of the payoffs, the ex-ante welfare of the individuals at a stationary equilibrium is given by

$$2(1-\beta)V = f(K) - \delta K.$$

Since the equilibrium capital stock is inefficiently small in all cases, $K \leq K^*$, by (3), $f'(K) - \delta \geq \beta^{-1} - 1 > 0$. Thus, any increase in capital accumulation turns immediately into a welfare improvement. The effects of fiscal and monetary policy on capital accumulation differ depending on the equilibrium that realizes. When the economy is in the liquidity trap, by the equilibrium system, (9), (10) and (11), it can be seen immediately that x does not affect any of the equilibrium variables, hence, open market operations are ineffective at this equilibrium. This is what is expected for a liquidity trap. Taxation, on the other hand, affects the Tobin's q and the capital accumulation, but not the nominal interest rate, which is always nil. By direct computation, it can be seen that both the Tobin's q and capital accumulation are increasing in taxation. When liquidity is unhindered, the economy may be in a regime with scarce or plentiful liquidity. Scarce liquidity in this context means that the stock of bonds is small relative to the stock of money, i.e. $x < 1$. This tightens the liquidity constraint since debt is used as an instrument to reallocate money before trade in the secondary market for capital occurs. Therefore, in this economy an expansionary monetary policy may induce a tightening of liquidity. In this case, the equilibrium prices and capital accumulation are pinned down implicitly by the equilibrium system (18), (19) and (20). Both fiscal and monetary policy affect all the equilibrium variables. By implicit differentiation, one can compute the effects of policy on the equilibrium prices and capital accumulation. Interestingly, for some parameters values, in particular, it can be checked that, when x is not too small and β is sufficiently close to 1, a contractionary open market operation may be optimal. Also with plentiful liquidity, i.e. $x \geq 1$, all the equilibrium variables depend both on monetary and fiscal policy as it can be seen from the equilibrium system, (23), (24) and (25). Specifically, the price of the bond and Tobin's q are increasing in fiscal policy and decreasing in the bonds to money ratio. In other words, there is a liquidity effect of open market operations. Also capital accumulation is increasing in fiscal policy, but decreasing in the bonds to money ratio, implying that an expansionary open market operation increases capital accumulation with beneficial effects for output and welfare.

7 Conclusion

We have presented a model in which three assets may be used for transaction purposes, money, debt and equity. We have shown that there are two types of equilibria. In the first type, money, debt and equity are all perfect substitutes as direct payment instruments to trade second hand capital. The interest rate on debt is nil, capital and output are inefficiently small, monetary policy is ineffective. In the second type of equilibrium, debt helps reshuffle misallocated money and equity may substitute for money or complement it. At this equilibrium, the interest rate on debt is positive, there may be an equity premium, capital and output are higher than in the previous equilibrium and monetary policy is effective. We have interpreted the first equilibrium as a liquidity trap and the second as a liquid situation in which liquidity may sometimes be scarce. Which equilibrium emerges depends on the coordination of the traders' expectations, but once they coordinate on the liquid equilibrium, the returns of the assets and capital accumulation depend on the availability of liquidity. In the liquid equilibrium, either a monetary expansion or contraction may be required to boost output and welfare, depending whether liquidity is plentiful or scarce.

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8 Appendix

In this Appendix, we derive the equilibrium conditions for the model and prove the Propositions in the text.

8.1 The Trap

Let us begin with the case in which the payment instruments are substitutes. Define $\gamma^E \geq 0$ the multiplier of constraint (5). The optimality condition for k^d is

$$f'\left(k + k^d\right) + 1 - \delta - q - \gamma^E q = 0. \tag{31}$$

Define $\gamma^I \geq 0$ the multiplier of constraint (6). The optimality condition for k^s is

$$q - 1 - \gamma^I = 0. \tag{32}$$

Denoting with $V_n(\cdot)$ the partial derivative wrt $n = m, b, k$, the optimality conditions for the assets holdings are:

$$v = \beta V_m\left(m_{+1}, b_{+1}, k_{+1}\right), \tag{33}$$

for money;

$$v\rho = \beta V_b\left(m_{+1}, b_{+1}, k_{+1}\right), \tag{34}$$

for bonds; and

$$1 = \beta V_k\left(m_{+1}, b_{+1}, k_{+1}\right), \tag{35}$$

for capital. The envelope conditions are

$$V_m(m, b, k) = v\left(1 + \frac{\gamma^E}{2}\right), \tag{36}$$

for money;

$$V_b(m, b, k) = v\left(1 + \frac{\gamma^E}{2}\right), \tag{37}$$

for bonds; and

$$V_k(m, b, k) = \frac{1}{2}\left[f'\left(k + k^d\right) + 1 - \delta + \gamma^E(1 - \delta) + \gamma^I + 1\right], \tag{38}$$

for capital. Insert the multipliers γ^E and γ^I obtained from (31) and (32) into (36), (37) and (38), delay them one period and combine them with (33), (34) and (35), obtaining the following optimality conditions: the Euler condition for money holdings

$$1 = \frac{\beta}{2}\frac{v_{+1}}{v}\left[\frac{f'(K_{+1}) + 1 - \delta}{q_{+1}} + 1\right], \tag{39}$$

for government bonds

$$\rho = \frac{\beta}{2}\frac{v_{+1}}{v}\left[\frac{f'(K_{+1}) + 1 - \delta}{q_{+1}} + 1\right], \tag{40}$$

for capital accumulation

$$1 = \frac{\beta}{2}\left\{f'(K_{+1}) + 1 - \delta + \left[\frac{f'(K_{+1}) + 1 - \delta}{q_{+1}} - 1\right](1 - \delta) + q_{+1}\right\}, \tag{41}$$

and the complementary slackness conditions for the constraint (5)

$$[f'(K) + 1 - \delta - q]\left[vm + vb + k(1 - \delta) - qk^d\right] = 0, \tag{42}$$

and for the constraint (6)

$$(q - 1)(k - k^s) = 0. \tag{43}$$

The first result is that, by arbitrage, the price of bonds must equal 1.

Lemma 1 *At an optimum,* $\rho = 1$.

Proof. Immediate by (39) and (40). ■

We consider equilibria in which the investors although indifferent between selling capital in the secondary market set $k^s = k$. Hence, (43) has been taken care of. The market clearing conditions are $k^s = k^d$, $m = M$, $b = B$, and the market clearing condition for the durable good, which holds by Walras Law whenever the other markets are in equilibrium. At a stationary equilibrium, the return of money is determined by

$$\frac{v_{+1}}{v} = \frac{1}{1 - \tau}, \tag{44}$$

where the RHS is given by the inverse of (1) with $\rho = 1$. The next result shows that the constraint (5) is binding at equilibrium.

Lemma 2 *At an equilibrium with trapped liquidity, (5) is binding.*

Proof. From (39) with (44) together with (41) at a stationary equilibrium, obtain

$$f'(K) + 1 - \delta = q\left[\frac{2(1-\tau) - \beta}{\beta}\right] \geq q$$

with a strict inequality iff $\tau < 1-\beta$. By (42), (5) is strictly binding if $f'(K)+1-\delta > q$. When $\tau = 1-\beta$, the constraint is just binding by continuity. ■

This reduces the equilibrium system to (9), (10) and (11) in the text. Next, we prove the Proposition in the text for this case.

Proof of Proposition 1. Solve (10) for

$$f'(K) + 1 - \delta = q\left[\frac{2(1-\tau) - \beta}{\beta}\right], \tag{45}$$

and plug it into (11), obtaining

$$F(q) \equiv (1-\tau)q - (1-\tau-\beta)(1-\delta) = 0. \tag{46}$$

By the linearity of $F(q)$ a solution of (46) exists and is unique. To guarantee $q \geq 1$, $\delta \geq 1 - \frac{\tau}{1-\tau-\beta} \equiv \widehat{\delta}$ is needed, where $\widehat{\delta} < 1$ provided $\tau > 0$. Once the solution for q is obtained (45) gives uniquely the equilibrium K.■

Proof of Proposition 2. The constraint on taxation is $\tau \leq \frac{(1-\delta)k}{vM(1+x)}$. At equilibrium, both (5) and (6) are binding. By the binding constraint (5) with (12), and the binding constraint (6) with market clearing in the secondary market for capital, obtain $\frac{k}{vM(1+x)} = \frac{1-\tau}{2\delta-1+2\tau(1-\delta)+\beta(1-\delta)}$.Substituting this into the constraint on taxation, obtain the upper bound on taxation. Verify that $1-\tau > \beta$ if $\delta > \frac{3\beta-1-\beta^2}{2\beta-\beta^2} \equiv \overline{\delta}$.■

8.2 The Flow

Consider now the case in which the liquidity is unhindered. Take the situation with scarce liquidity first.

8.2.1 Scarce Liquidity

The multipliers are $\lambda^E \geq 0$ and $\theta^E \geq 0$ for (14) and (15), respectively. The optimality condition for k^d is

$$f'\left(k + k^d\right) + 1 - \delta - q - \theta^E q = 0, \tag{47}$$

and for m^d

$$v - p - \lambda^E p + \theta^E v = 0. \tag{48}$$

The multipliers are $\lambda^I \geq 0$ and $\theta^I \geq 0$ for (16) and (17), respectively. The optimality condition for k^s is

$$q - 1 - \theta^I = 0, \tag{49}$$

and for m^s

$$p - v - \lambda^I v = 0. \tag{50}$$

The optimality conditions for the assets holdings are (33), (34) and (35). The envelope conditions are

$$V_m (m, b, k) = v \left(1 + \frac{\theta^E + \lambda^I}{2}\right), \tag{51}$$

for money;

$$V_b (m, b, k) = v \left(1 + \frac{\lambda^E}{2}\right), \tag{52}$$

for bonds; and

$$V_k (m, b, k) = \frac{1}{2} \left[f' \left(k + k^d\right) + 1 - \delta + \lambda^E (1 - \delta) + \theta^I + 1\right], \tag{53}$$

for capital. Insert the multipliers λ^E, θ^E, λ^I and θ^I obtained from (47), (48), (49) and (50) into (51), (52) and (53), delay them one period and combine them with (33), (34) and (35), obtaining the following optimality conditions: the Euler conditions for money holdings

$$1 = \frac{\beta}{2} \frac{v_{+1}}{v} \left[\frac{f' (K_{+1}) + 1 - \delta}{q_{+1}} + \frac{p_{+1}}{v_{+1}}\right], \tag{54}$$

for government bonds

$$\rho = \frac{\beta}{2} \frac{v_{+1}}{v} \frac{v_{+1}}{p_{+1}} \left[\frac{f' (K_{+1}) + 1 - \delta}{q_{+1}} + \frac{p_{+1}}{v_{+1}}\right], \tag{55}$$

for capital accumulation

$$1 = \frac{\beta}{2} \left\{f' (K_{+1}) + 1 - \delta + q_{+1} + \left[\frac{f' (K_{+1}) + 1 - \delta}{q_{+1}} \frac{v_{+1}}{p_{+1}} - 1\right] (1 - \delta)\right\}, \tag{56}$$

and the complementary slackness conditions for the constraint (14)

$$\left[f'(K)+1-\delta-q\rho^{-1}\right]\left[vb+(1-\delta)k-pm^{d}\right]=0, \tag{57}$$

for the constraint (15)

$$\left[f'(K)+1-\delta-q\right]\left[vm+vm^{d}-qk^{d}\right]=0, \tag{58}$$

for the constraint (16)

$$(p-v)\,v\,(m-m^{s})=0, \tag{59}$$

and for the constraint (17), which is

$$(q-1)(k-k^{s})=0. \tag{60}$$

The first result follows from arbitrage and equates the price of bonds to the relative price of money in the liquidity and primary markets.

Lemma 3 *At an optimum, $\rho=\frac{v}{p}$.*

Proof. Immediate by (54) and (55). ∎

By this Lemma, among the Euler conditions we only need to check (54) and (56). We look for equilibria in which $m^{s}=m$ even when $p=v$ and $k^{s}=k$ even when $q=1$. The next Lemma simplifies the equilibrium system, under this assumption.

Lemma 4 *Constraint (14) implies (15).*

Proof. Since $\rho^{-1}\geq 1$, $f'(K)+1-\delta\geq q\rho^{-1}\geq q$, hence, by (58), (15) is implied by (14). ∎

Therefore, among the complementary slackness conditions we only need to check (57). Next, we show that for $x<1$ and δ not too small, also (14) is binding. This is required for both debt and equity to be used in the LM to reshuffle liquidity.

Lemma 5 *If $x<1$ and $\delta>\frac{1+x}{2}$, the constraint (14) is binding.*

Proof. Use the binding (15) to write (14) as $\rho^{-1}q\leq qx+2(1-\delta)$. since $q\geq 1$ and $\rho^{-1}q\geq 1$, the constraint is always binding when $x+2(1-\delta)<1$. ∎

The market clearing conditions are $k^s = k^d$, $m^s = m^d$, $m = M$, $b = B$, and the market clearing condition for the durable good, which holds by Walras Law whenever the other markets are in equilibrium. At a stationary equilibrium, the return of money is determined by

$$\frac{v_{+1}}{v} = \frac{1+\rho x}{(1-\tau)(1+x)}, \tag{61}$$

where the RHS is given by the inverse of (1). Using the binding constraints and the market clearing conditions, we obtain the equilibrium conditions (18)-(20) in the text. Next, we prove the Proposition in the text for this case.

Proof of Proposition 3. Write (18) as

$$f'(K) + 1 - \delta = q\rho^{-1}\frac{2(1-\tau)(1+x)\rho - \beta(1+\rho x)}{\beta(1+\rho x)}, \tag{62}$$

plug it into (18) and write it as

$$q = \rho\frac{2}{\beta}\frac{(1+\rho x)(1+\beta-\beta\delta) - (1-\delta)(1-\tau)(1+x)\rho}{2(1-\tau)(1+x)\rho - \beta(1+\rho x)(1-\rho)}. \tag{63}$$

Use (62) and (63) into (20), obtaining

$$G(\rho) \equiv \rho^2 x A + \rho(1-\delta)B - C = 0. \tag{64}$$

where $A \equiv x - (1-\delta)(1-\tau-\beta)(1+x)$, $B \equiv 3(1-\tau)(1+x) + \beta(1-x)$ and $C \equiv 1 + 2\beta(1-\delta)$. If $\delta \geq 1 - \frac{x}{(1-\tau-\beta)(1+x)} \equiv \underline{\delta}$, (64) admits a positive solution with $\rho \leq 1$ provided $x \geq \frac{(3\tau+\beta)(1-\delta)+3\delta-2}{(\tau+\beta)(1-\delta)+\delta} \equiv \underline{x}$, where $\underline{x} < 1$ since both $\delta < 1$ and $\tau < 1$. The positive solution is unique. Once ρ is determined, (62) and (63) pin down uniquely q and K.■

8.2.2 Plentiful Liquidity

The multipliers are $\sigma^E \geq 0$ and $\zeta^E \geq 0$ for (21) and (22), respectively. The optimality condition for k^d is

$$f'\left(k + k^d\right) + 1 - \delta - q - \zeta^E q = 0, \tag{65}$$

and for m^d

$$v - p - \sigma^E p + \zeta^E v = 0. \tag{66}$$

The optimality conditions for the investor are the same as in the scarce liquidity case.The optimality conditions for the assets holdings are (33), (34) and (35). The envelope conditions are

$$V_m(m,b,k) = v\left(1 + \frac{\zeta^E + \lambda^I}{2}\right), \tag{67}$$

for money;

$$V_b(m,b,k) = v\left(1 + \frac{\sigma^E}{2}\right), \tag{68}$$

for bonds; and

$$V_k(m,b,k) = \frac{1}{2}\left[f'\left(k + k^d\right) + 1 - \delta + \zeta^E(1-\delta) + \theta^I + 1\right], \tag{69}$$

for capital. Insert the multipliers σ^E, ζ^E, λ^I and θ^I obtained from (65), (66), (49) and (50) into (67), (68) and (69), delay them one period and combine them with (33), (34) and (35), obtaining the following optimality conditions: the Euler conditions for money holdings, (54), for government bonds, (55), for capital accumulation, (56) and the complementary slackness conditions for the constraint (21)

$$\left[f'(K) + 1 - \delta - q\rho^{-1}\right]\left[vb - pm^d\right] = 0, \tag{70}$$

for the constraint (22)

$$\left[f'(K) + 1 - \delta - q\right]\left[vm + vm^d + (1-\delta)k - qk^d\right] = 0, \tag{71}$$

for the constraint (16), (59) and for the constraint (17), (60). Lemma 3 holds here as well for the same reason, hence, $\rho = \frac{v}{p}$.Thus, we only need to check (54) and (56). We look for equilibria in which $m^s = m$ even when $p = v$ and $k^s = k$ even when $q = 1$. The next Lemma simplifies the equilibrium system, under this assumption.

Lemma 6 *Constraint (21) implies (22).*

Proof. Since $\rho^{-1} \geq 1$, $f'(K) + 1 - \delta \geq q\rho^{-1} \geq q$, hence, by (71), (22) is implied by (21). ■

Therefore, among the complementary slackness conditions we only need to check (70). The market clearing conditions are $k^s = k^d$, $m^s = m^d$, $m = M$, $b = B$, and the market clearing condition for the durable good, which holds by Walras Law whenever the other markets are in equilibrium. At a stationary equilibrium, the return of money is determined by (61). Using the binding constraints and the market clearing conditions, we obtain the equilibrium conditions (23)-(25) in the text. Next, we prove the Propositions in the text for this case.

Proof of Proposition 4. From (23) and (24) obtain

$$\frac{f'(K)+1-\delta}{q\rho^{-1}} = \frac{2(1+x)(1-\tau)\rho}{1+\rho x} - 1. \tag{72}$$

Insert (72) into (25), obtaining

$$\{[1-\tau+(1-\tau-\beta)x]\rho-\beta\}(\rho x-1)=0. \tag{73}$$

Since both terms in brackets are linear in ρ, a unique $\widetilde{\rho}$ exists that satisfies (73), whether the constraint is binding or not, with $\widetilde{\rho} \leq 1$ provided $x \geq 1$. Once $\widetilde{\rho}$ is determined (23) and (24) give the other equilibrium variables uniquely.■

Proof of Proposition 5. Substitute (29) into (70), obtaining that the constraint is slack iff $x \geq \frac{1-\tau}{2\beta-(1-\tau)} \equiv \overline{x}$, with $\overline{x} > 1$ since $\beta < 1$.■

Proof of Proposition 6. By (13) and (28), obtain $\widetilde{K} > \widehat{K}$, when

$$x > \frac{\beta-(1-\delta)(1-\tau)^2-(1-\delta)(1-\tau-\beta)\beta}{(1-\delta)(1-\tau-\beta)\beta} \equiv \widetilde{x}.$$

By (4), $K^* > \widetilde{K}$, for $\tau < 1-\beta$. When $\tau = 1-\beta$, $K^* = \widetilde{K} = \widehat{K}$. At equilibrium, the constraint (22) is binding. Substituting (27) in it, obtain

$$\frac{k}{vM} = \frac{2\beta+(1+x)(1-\tau-\beta)}{\delta+(1-\delta)\tau-(1-\delta)(1+x)(1-\tau-\beta)}.$$

Using this into the constraint on taxation, $\tau \leq \frac{(1-\delta)k}{vM(1+x)}$, it is immediate to verify that $1-\tau > \beta$ if $\delta > \frac{2\beta-(1-\beta)^2(1+x)}{2\beta+\beta(1-\beta)(1+x)}$, which is smaller than $\overline{\delta}$.■

8.2.3 Sunspot Equilibrium

Finally, we prove the Proposition for the existence of sunspot equilibria.

Proof of Proposition 7. Take the limit $\phi_{ss'} = 0$ with $s = s'$ for both s, thus, approximating a deterministic switch between states every period. The equations (30) become

$$q_s f'^{-1}\left(\frac{2-\beta+\beta\delta-\beta q_s}{\beta}\right)=\frac{\beta}{2}\frac{1}{\pi_{s'}}\left[f'^{-1}\left(\frac{2-\beta+\beta\delta-\beta q_{s'}}{\beta}\right)+q_{s'}\sigma_{s'}\right], \quad (74)$$

for each s, with $s'\neq s$. Define $h(q)\equiv qf'^{-1}\left(\frac{2-\beta+\beta\delta-\beta q}{\beta}\right)$, which is monotonically increasing in q, hence, invertible. From (74) with $s=1$, solve for

$$q_1=h^{-1}\left(\frac{\beta}{2}f'^{-1}\left(\frac{2-\beta+\beta\delta-\beta q_2}{\beta}\right)+\frac{\beta}{2}q_2\right), \quad (75)$$

and plug it into (74) with $s=2$, obtaining

$$H(q_2)\equiv 2\left[1+(1-\beta)x\right]h(q_2)-\beta f'^{-1}\left(h^{-1}(q_2)\right)-\left[1+(1-\beta)x\right]h^{-1}(q_2)=0. \quad (76)$$

Since $H(q_2)$ is continuous in q_2 and $H(1)H\left(\beta^{-1}\right)<0$, by the Intermediate Value Theorem a q_2 such that $H(q_2)=0$ exists for this case. Once q_2 is pinned down (75) gives q_1 uniquely. The requirement $q_1\neq q_2$ is satisfied since $\pi_1\neq\pi_2$ and $\sigma_1\neq\sigma_2$. By continuity, a stationary sunspot equilibrium exists also for probabilities $\phi_{ss'}$ with $s=s'$ for both s close to but bounded away from zero.■