Central-Bank Digital Currency and Risk Sharing in a Currency Union*

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February 22, 2021

Abstract

In this paper we construct a two-country model of monetary exchange and bank credit to study resource allocation and risk sharing within a currency union. We study the market equilibrium that prevails when there is perfect integration of cross-border credit markets and when credit markets are fragmented, which prevents banks from engaging in cross-border lending activities. We find that credit market fragmentation yields a suboptimal allocation, both in terms of capital allocation and consumption risk sharing. We use this framework to study the implications of two supra-national policies when credit markets are fragmented: a common deposit insurance and a central-bank digital currency. While both policies improve consumption risk sharing, they differ in their effects on capital allocation.

1 Introduction

The desirable level of financial and banking integration among countries of the European Monetary Union has recently been at the heart of policy discussions. The banking union agenda has emerged as a response to highly concentrated national sovereign debt exposure and insufficient cross-border financial activity among member states, to promote monetary and financial integration in the union.[1] One component of the banking union is the proposition of a common deposit insurance, that would be set at the federal level, as recently advocated by Germany's minister of finance Olaf Scholz. At the same time, the European Central Bank is analysing the issuance of a central-bank digital currency.[2] The digital euro would ensure to all citizens of the euro zone equal access to a risk-free means of payment, easier to use in a larger number of transactions than physical cash, thereby promoting monetary integration.

*Financial support from ANR (project ANR-18-CE26-0001-01) is gratefully acknowledged.

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[1] For example, cross-border lending to non financial corporations represented only 6% of total loans in 2019 [2] and is seen as inefficiently low[3].

[2] European Central Bank. (2020). Report on a digital euro. European Central Bank.

These policies are expected to have an impact on resource allocation and risk sharing among the countries that belong to the euro zone, but their precise effects are still to be studied. In this paper, we provide a formal framework to study how these policies affect capital allocation, consumption and welfare within a monetary union. To create a risk sharing motive across countries, we consider idiosyncratic productivity shocks that occur at the country-level. The banking sector has two roles since it provides consumers with a means of payment for transactions in the goods market, and provides credit to firms. We study resource allocation in this economy by explicitly considering a retail credit market that is integrated across the borders or, alternatively, fragmented along the national frontiers.

More precisely, we construct a two-country model of monetary exchange with banks that issue deposits and grant loans to entrepreneurs. In order to acquire bank deposits, consumers need to deposit physical currency with their national banks. Banks extend loans to entrepreneurs who use the borrowed cash to buy capital from suppliers. To capture the idiosyncratic country risk, we assume that every period a random negative productivity shock hits more severely one of the two countries, implying that a fraction of the entrepreneurs in this country will fail to produce. This shock then affects bank profitability and hence the level of deposit interest rates, which in turn impacts depositors' welfare.

To have a benchmark to assess the market allocation, we first consider a social planner that gives equal weight to citizens in both countries and sets optimal levels of capital allocation and consumption. As a market benchmark for our fragmented banking scenario, we study the market equilibrium that prevails under perfect integration of cross-border credit markets. We then consider a setting where credit markets are fragmented and banks are unable to engage in cross-border lending activities.

First, we show that perfect integration of the credit markets allows replicating the social planner solution if monetary policy sets the cost of holding money balances to zero; i.e., it follows the Friedman rule. If the Friedman rule is not implementable, then it is optimal to run a monetary policy that sets a low inflation rate. Integration of credit markets allow banks to adjust their portfolio by lending to both *ex ante* risky and riskless firms. As a result, in equilibrium bank deposits issued by banks in the two countries offer the same return. Consumers are therefore able to perfectly smooth consumption across periods, regardless of the realization of the shock in their country.

When credit markets are segmented at the country level, the aggregate level of invested capital is lower than what is socially optimal. We find that there is a threshold value of the idiosyncratic productivity shock that characterizes capital allocation in the currency union. If the technology in the risky country is not very productive, the level of capital in the risky country is lower than in the country with no risk. However, if technology in the risky country is sufficiently productive, being still less productive than in the safe country, the level of invested capital is the same in both countries. This results in overinvestment in the risky country and underinvestment in the safe country compared to the social optimum. With fragmented credit markets, consumption risk sharing is inefficient, since remuneration on consumers' bank deposits depends on the realization of aggregate risk.

We pursue our analysis of the fragmented banking setting to study how a common deposit insurance, shared by the two countries, would alter the above results. We find that deposit insurance does not modify lending rates when consumers are unconstrained in the goods market. However since a common deposit insurance allows depositors to smooth consumption over the states of nature, consumers are able to attain their optimal consumption level for larger risk levels than in the fragmented banking equilibrium with no deposit insurance. In this setting, the deposit insurance does not improve the allocation of capital across countries of the currency union. The reason is that credit is more expensive for the firms located in the less risky country, which reduce their demand for capital. In turn, the firms located in the risky country face lower prices and can increase their capital investment. The deposit insurance may then exacerbate distortions in the capital allocation due to the fragmentation of the banking markets.

Finally, we present an extension of the model where we introduce a digital currency issued by the monetary authority, equally available to consumers of the two countries. This digital currency is then a risk-free means of payment, and therefore allows consumers to perfectly smooth consumption over time. We show that, unlike a common of deposit insurance, by making capital investment more expensive a central-bank digital currency attenuates distortions in the allocation of capital and is therefore unambiguously welfare improving.

This paper relates to the literature on optimal currency areas that seeks to identify conditions for currency unions to be optimal [12, 10]. A large number of papers in this field relate to the role played by the interaction between monetary and fiscal policies in the optimality of monetary unification [15, 9, 7, 14]. This paper focuses on another dimension, the degree of integration of credit markets among countries that form a currency union. Using a similar framework, [6] study how, by reducing currency transaction costs, a currency union may affect borrowing constraints and thereby the level of credit supported in equilibrium. This paper also aims to contribute to the recent literature on central-bank digital currency [1, 8, 4]. Our paper explores a specific dimension, the effects of issuing a central-bank digital currency in the context of a currency union with a motive for risk sharing among countries.

Our paper proceeds as follows. We first describe the environment and planner solution in Section 2. Section 3 presents the agents' decision problems and equilibria when banking is fragmented. In Section 4 we derive equilibria when banking is fragmented. Section 5 studies the effects of common deposit insurance on the equilibria and compares with the fragmented banking equilibrium. Section 6 introduces Central Bank Digital Currency and its effects on allocations compared to fragmented banking. The last section concludes.

2 Environment

The environment builds in the sequential-market model developed by Lagos and Wright [11] and Rocheteau and Wright[13]. Time is discrete and lasts forever. Each period is divided into three sequential markets: a *capital market*, a *goods market* and a *settlement market*. Time across periods is discounted with factor $\beta \in (0, 1)$. The economy is composed of 2 identical countries: H (home) and F (foreign). Each country has four types of agents, each with unit measure: infinitely lived *suppliers, buyers, sellers* and one period lived *entrepreneurs*. Agent types are permanent.

Suppliers can produce capital k in the capital market at cost $c(k)$ with $c'(k), c''(k) > 0$. Capital is storable intra period but not across periods. Entrepreneurs are one period lived and are born with no endowment. They own a technology that allows them to invest capital to produce $f(k)$ units of settlement-market goods, with $f'(k) > 0$, $f''(k) < 0$, $f'(0) = \infty$ and $f'(\infty) = 0$.

Every period, each country $i = h, f$ is subject to an aggregate productivity shock θ_i. We assume that θ_h and θ_f are inversely correlated in order to focus on risk sharing across countries.[3] An entrepreneur who invests k produces nothing with probability θ and $f(k)$ with probability $1-\theta$. We will refer to the country with a positive firm default rate θ as *risky*, and to the other as *safe*. We assume that there are two states of nature $\{\theta_h, \theta_f\}$, which are named after the risky country:

	Country h	Country f
θ_h	θ_h	0
θ_f	0	θ_f

Let σ be the probability of state θ_h and $1-\sigma$ of being in θ_f. To keep the symmetry in the baseline model we set $\sigma = 1/2$ and $\theta_h = \theta_f = \theta$.

In the goods market, buyers get utility $u(q)$ from consuming q goods, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$. Sellers can produce q at linear cost. In the settlement market, buyers and sellers can produce and consume with linear utility (a negative consumption quantity is interpreted as production). Suppliers and entrepreneurs obtain linear utility from consumption.

In every country there are competitive *banks*, similar to [5]. Banks accept nominal fiat money in exchange for nominal electronic deposits. Banks can lend the deposited cash to entrepreneurs. Banks create money in our model by accepting fiat money from buyers and issuing deposits in exchange while making cash loans to entrepreneurs. We assume banks have a record keeping technology to keep track of credit histories and electronic deposit transactions at no cost. Financial contracts are one-period contracts.[4] We assume full commitment in financial relationships, thus agents have no strategic default option.

We make the following assumptions regarding transactions. Transactions in the goods market can only be carried out with an electronic means of payment; i.e., bank deposits.[5] Thus, buyers deposit fiat money balances in banks in order to obtain deposits. To create a motive for the use of fiat money, we assume capital market transactions

[3]This modelling choice allows us to capture the idiosyncratic risk of countries. We could have a more general productivity shock. What is important for our results is that every period one country is riskier or safer than the other.

[4]Linear disutility in the settlement market coupled with the fact that agents make the same decision every period makes it unnecessary to rollover loan contracts.

[5]More generally, we could have a fraction of transactions that can be carried out with any means

require immediate settlement.
The central bank issues fiat money at a money growth rate $\gamma = M/M_{-1}$, where M is the money stock. Newly-issued money is used to make lump-sum transfers in the settlement market.

The timing of a period is as follows. The state of nature of the economy is revealed to all agents at the beginning of the period. The capital market opens and entrepreneurs take out cash loans from banks to buy capital from suppliers who produce on the spot. We assume capital transactions require immediate settlement which requires physical fiat currency payment. We assume the capital market is perfectly integrated, so entrepreneurs can purchase capital from foreign or home suppliers with no difference. Then the goods market opens and buyers who deposited physical fiat currency in the previous period can purchase goods from sellers. We assume that goods market are country specific. Then the settlement market opens and all agents consume. Firms sell their production and repay loans to banks who redeem deposits. Buyers choose their portfolios for the next period (work to obtain physical fiat money and deposit it).

A key aspect of the paper is the *fragmentation* of the banking market. The banking market is subdivided into two activities: deposits and loans. We consider that the loan market is fragmented if entrepreneurs cannot take out loans from foreign banks. If banks can lend to any entrepreneurs, then we consider that the loan market is *integrated*. We will also study the implications of fragmentation for the deposit market, i.e. if buyers can deposit money in foreign banks or not.

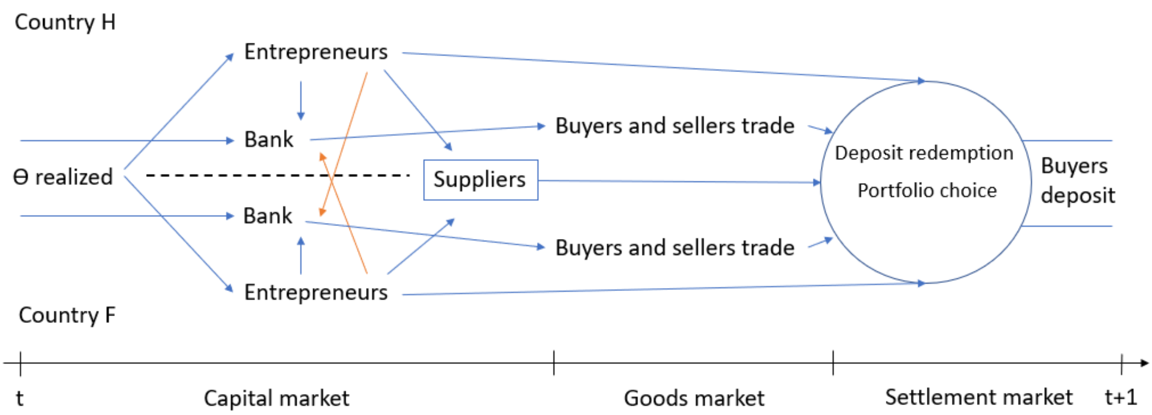

Figure 1: Market timing

2.1 Efficient allocation

In order to have a benchmark for capital allocation and welfare, consider a social planner who has control over both countries and assigns equal weight to all agents. The social planner is assumed to have the ability of reallocating resources in the

of payment, and a fraction that can be only carried out with an electronic means of payment, for example online transactions. Our choice is made for simplicity.

economy whenever the shock occurs. Since we study the symmetric case and shocks are inversely correlated, the social planner knows that one country will be risky and the other one safe[6]. Thus, social welfare is:

$$(1-\beta)\mathcal{W} = u\left(q^{\theta}\right) + u\left(q\right) - q_s^{\theta} - q_s - \\ c\left(k^{\theta}\right) - c\left(k\right) + (1-\theta)\, f\left(K^{\theta}\right) + f\left(K\right)$$

where k (k^{θ}) is the quantity of capital produced by suppliers in the safe (risky) country and K (K^{θ}) is the total capital invested in the safe (risky) country. Consumption by buyers in the safe and risky country are respectively denoted by q and q^{θ}. Production by sellers in safe (risky) country is denoted by q_s (q_s^{θ}). The planner's choice is subject to the capital constraint

$$k + k^{\theta} \geq K + K^{\theta}$$

so resources can be perfectly reallocated across countries. The social planner maximizes social welfare subject to the capital constraint. It is straightforward to see that optimal consumption by buyers satisfy $u'\left(q\right) = u'\left(q^{\theta}\right) = 1$, with $q^{\theta} + q = q_s + q_s^{\theta}$. The following Lemma defines the efficient allocation.

Lemma 1 *Optimal capital allocation is* $\left\{k, k^{\theta}, K, K^{\theta}\right\}$ *that solves* $k = k^{\theta} = k^{\star}$, $k^{\star} = \left(K + K^{\theta}\right)/2$ *and*

$$(1-\theta)f'(K^{\theta}) = c'(k^{\star}) \quad (1)$$

$$f'(K) = c'(k^{\star}) \quad (2)$$

If $\theta > 0$*, then the optimal allocation satisfies* $K > k^{\star} > K^{\theta}$.

Proof From (1) and (2), if $\theta > 0$ the optimal capital allocation satisfies $f'\left(K\right) < f'\left(K^{\theta}\right)$ and hence $K > K^{\theta}$. Since the capital constraint rewrites $2k^{\star} = K + K^{\theta}$, it is straightforward that $K > k^{\star} > K^{\theta}$.

Since production is more efficient in the safe country, it is optimal to transfer capital produced by suppliers in the risky one to the safe one.

3 Integrated banking

Let us first consider the market economy described above, with no deposit insurance or CBDC. In this section we make the assumption that loan markets are perfectly integrated; i.e., there is a cross-border loan market which allows firms to take out loans from foreign banks. We solve the model backwards, starting from the settlement market, for a representative period.

[6] Since we assume $\theta_h = \theta_f = \theta$, welfare when country h is hit by the shock is the same than the one when country f is hit, i.e. $W_{\theta_h} = W_{\theta_f} = W$. Thus, there is no overall uncertainty since $\mathbb{E}_{\theta}\left[W\right] = \frac{1}{2}W_{\theta_h} + \frac{1}{2}W_{\theta_f} = W$

3.1 Settlement market

Agents enter the settlement market after trading in the goods market. The current period state of nature is known by all agents and banks. We have to consider the program agents follow in both states of nature and solve the model backwards.

3.1.1 Portfolio choice

Buyers deposit in the settlement market, consume x, work h and redeem previous deposits with interest. Let us model the portolio choice (m, d) of a buyer. Let $W(m, d)$ and $W^\theta(m, d)$ respectively be the value of entering the settlement market with m nominal money balances and d nominal deposit balances in the safe and risky state. An agent in the risky state[7] solves:

$$W^\theta(m,d) = \max_{x, m_{+1}, d_{+1}} x + \beta \left[\frac{1}{2} V(m_{+1}, d_{+1}) + \frac{1}{2} V^\theta(m_{+1}, d_{+1}) \right]$$

subject to

$$x + \phi^\theta(m_{+1} + d_{+1}) = h + \phi^\theta(1 + i_d^\theta)d + \phi^\theta m$$

where ϕ^θ is the price of money in terms of settlement market goods and i_d^θ the interest rate on deposits in the risky country.

Agents reason in expected value in the settlement market since the shock occurs at the beginning of the capital market in the following period. The first order conditions are $u'(x) = 1$, $\beta V_m(m, d) = \phi_{-1}$, $\beta V_m(m, d) = \phi_{-1}^\theta$ and:

$$\beta \left[\frac{1}{2} V_d(m,d) + \frac{1}{2} V_d^\theta(m,d) \right] = \phi_{-1}$$

$$\beta \left[\frac{1}{2} V_d(m,d) + \frac{1}{2} V_d^\theta(m,d) \right] = \phi_{-1}^\theta$$

from which we can deduce that the price of money in terms of CM goods is the same independently of the state the economy is in such that $\phi_{-1}^\theta = \phi_{-1}$. Since we study stationary monetary equilibria, i.e. $\phi_{-1}^\theta / \phi^\theta = \phi_{-1}/\phi = \gamma$ we can deduce:

$$\phi^\theta = \phi \tag{3}$$

which is explained by the fact that agents are ex-ante symmetrical. Agents reason in expected value and decide on their portfolio with no possible readjustments ex-post. Money holdings are degenerate (i.e. do not depend on the state in the previous period). This allows us to rewrite the first order conditions on deposit holdings as:

$$\frac{\beta}{2} \left[V_d(m,d) + V_d^\theta(m,d) \right] = \phi_{-1} \tag{4}$$

We also obtain the following envelope conditions:

$$W_d^\theta = \phi(1 + i_d^\theta) \tag{5}$$

$$W_d = \phi(1 + i_d) \tag{6}$$

$W_m = \phi^\theta$ and $W_m^\theta = \phi$.

[7] The program for an agent in the safe state is closely the same except for the value of money ϕ and the interest rate on deposits i_d which can be different.

3.2 Goods market

Agents are perfectly informed of the state of nature in the market for goods and therefore know which will be their value of entering in the settlement market after. For simplicity, we formally describe the programs of agents in the risky state as the one in the safe state hardly differs.

3.2.1 Buyers

Prices are set competitively. Let the value of being a buyer entering the goods market with portfolio (m, d) in the good and bad state be $V(m, d)$ and $V^\theta(m, d)$, respectively. Thus, a buyer in the bad state follows:

$$V^\theta(m, d) = \max_{q^\theta, q^\theta_m, q^\theta_d} u(q^\theta) + W^\theta(m - p^m q^\theta_m, d - p^\theta q^\theta_d)$$

subject to

$$m \geq p^m q^\theta_m$$
$$d \geq p^\theta q^\theta_d$$
$$q^\theta_d + q^\theta_m = q^\theta$$

where p^θ is the nominal price of deposits in the risky state. Consumption in the risky state q^θ is subdivided into q^θ_m and q^θ_d which denote consumption financed by money and deposits, respectively. Using (3), we obtain the following first order conditions on goods market consumption (assuming $m = 0$):

$$u'(q^\theta) \geq p^\theta \phi(1 + i^\theta_d)$$

and alternatively for the good state we get:

$$u'(q) \geq p\phi(1 + i_d)$$

where p is the nominal price of deposits and i_d the interest rate on deposits in the safe state.

3.2.2 Sellers

Sellers work with linear disutility and can accept both means of payment. The notation is the same type as the one used for buyers, where q^θ_s denotes the quantity sold by sellers in the risky state. Thus, a seller in the risky state solves:

$$\max_{q^\theta_s, q^\theta_{s,d}, q^\theta_{s,m}} -q^\theta_s + W^\theta(m + p^m q^\theta_{s,m}, d - p^\theta q^\theta_{s,d})$$

subject to

$$q^\theta_s = q^\theta_{s,m} + q^\theta_{s,d}$$

where q^θ_s is subdivided into the quantity produced by sellers sold for money $q^\theta_{s,m}$ and the one sold for deposits $q^\theta_{s,d}$. Using (6) and (3) we obtain the following first order conditions on quantities produced by sellers in both states:

$$p^\theta \phi(1 + i^\theta_d) = 1 \qquad (7)$$

$$p\phi(1+i_d)=1 \tag{8}$$

and the following on production sold for fiat money:

$$p_m\phi=1 \tag{9}$$

which simply reflects that the marginal cost of producing one unit of goods is equal to the marginal utility of consuming in the settlement market.

3.3 Capital market

3.3.1 Suppliers

Suppliers produce capital at convex cost and sell their production to firms who pay with the cash loans they obtained from banks. The product of the sale is then used to consume in the settlement market. Since we assume the capital market is integrated, sellers can symmetrically sell their production to foreign or home entrepreneurs. Thus, capital suppliers behave the same way, irrespective of their country and follow:

$$\max_{k_s} -c(k_s)+\phi p_k k^s \tag{10}$$

where p_k is the nominal price of capital, and does not depend on where it is sold. We obtain the following first order conditions on capital supplied by sellers in both states:

$$c'(k_s)=\phi p_k \tag{11}$$

where the left term is simply the marginal cost of capital supplied and the right side is the marginal utility of consuming in the settlement market with the fiat money balances obtained.

3.3.2 Entrepreneurs

Since loan markets are assumed to be integrated, then banks will charge the same interest rate on loans. Thus, we do not differentiate loans granted by banks in the risky and safe country. An entrepreneur in the risky state solves:

$$\max_{\ell^\theta,k^\theta} f(k^\theta)-(1+i^\theta)\phi\ell^\theta$$

subject to

$$p_k k^\theta \leq \ell^\theta$$

Notice that an entrepreneur does not take into account his risk of default. It is straightforward to see that firms will choose the volume of loans such that $p_k k^\theta=\ell^\theta$. Using (11), we obtain the following first order conditions on capital demand:

$$f'\left(k^\theta\right)=c'\left(k_s\right)\left(1+i^\theta\right) \tag{12}$$

$$f'\left(k\right)=c'\left(k_s\right)\left(1+i\right) \tag{13}$$

where the marginal productivity of capital is equal to the marginal cost.

Since suppliers sell their capital to entrepreneurs in both countries, the market for capital clear for:

$$2k_s = k + k^\theta \tag{14}$$

where the left side is the sum of the capital supplied by sellers from both countries. Remember that since capital markets are integrated, suppliers produce the same amount independently of where they are located. The right side is simply the sum of the capital purchased by entrepreneurs to produce settlement market goods.

3.4 Banks

We consider integrated banking markets. Banks can engage in domestic and cross-border lending with no difference. Banks have the same quality of information on the default rate of firms θ. Thus, we abstract from possible foreign firm quality assessment costs or uncertainty. We assume however that agents can only deposit in their national country. Thus, deposit markets are fragmented. Banks acquire deposits during the settlement period and redeem those deposits the following settlement market. Banks having the same investment possibilities, they solve the same following program:

$$\max_d (1+i)\ell + (1-\theta)(1+i^\theta)\ell^\theta - (1+i_d)d$$

subject to

$$\ell + \ell^\theta \leq d$$

where ℓ^θ (ℓ) and i^θ (i) are respectively the amount of loans and interest rate offered to safe (risky) firms. The first order condition then yields:

$$1 + i_d = \lambda$$

where λ is the Lagrangian multiplier on the bank's resource constraint. It is straightforward to see that $\lambda > 0$ since otherwise $i_d < 0$ which is not feasible since agents would not deposit money balances to earn a negative interest rate when they can just keep fiat money balances. It follows that banks lend all deposits, i.e. $d = \ell + \ell^\theta$.

Banks then choose the level of loans during the capital market after the productivity shock is revealed. Since depositors face the same uncertainty ex-ante the level of d is the same in both countries. Recall that banks face the same investment opportunities since cross-border banking is allowed with no frictions. Thus, banks from both countries follow the same program:

$$\max_{\ell,\ell^\theta} (1+i)\ell + (1-\theta)(1+i^\theta)\ell^\theta - (1+i_d)d$$

subject to

$$\ell + \ell^\theta = d$$

which yields the following indifference for banks:

$$(1+i) = (1-\theta)\left(1+i^\theta\right) \tag{15}$$

where the expected interest rate on loans to entrepreneurs with a positive default rate has to be equal to the one for safe entrepreneurs. If we had $(1+i) > (1-\theta)\left(1+i^{\theta}\right)$, then banks would lend to safe entrepreneurs only. This extreme case is prevented by our assumption $f'(0) = \infty$.

We suppose banks are perfectly competitive and make zero profits in every state. The zero profit condition is:

$$(1-\theta)(1+i^{\theta})\ell^{\theta} + (1+i)\ell - (1+i_d)d = 0$$

Since $\ell + \ell^{\theta} = d$, zero profit implies:

$$i = i_d \tag{16}$$

where $i_d \geq 0$ otherwise agents would rather take fiat money into the next period.

3.5 Stationary Integrated Banking Equilibrium

Countries are *ex ante* perfectly symmetric. We consider that banking is integrated, thus banks can engage in cross-border lending. Remember that we consider stationary monetary equilibria so the real money stock is time invariant:

$$\phi M = \phi_{-1} M_{-1} \tag{17}$$

Since banks have the same investment opportunities, we have $i_d^{\theta} = i_d$, so that agents get the same interest rate on deposits whatever their location is. Thus from (7)-(8) the nominal price of deposits is the same, i.e. $p^{\theta} = p$, and it follows that $q^{\theta} = q$. Using (17), (4) rewrites:

$$\beta u'(q) = \gamma/(1+i_d) \tag{18}$$

where the left term is the marginal utility of consuming in the following goods market. The right term reflects the marginal cost of bringing a unit of money in the following period. So agents bring deposits into the next period until it does not provide anymore benefit.

3.5.1 Unconstrained equilibrium

In an unconstrained equilibrium, buyers consume such that $u'(q) = 1$. From (18), and using (3) the expected return on deposits must follow:

$$i_d = \gamma/\beta - 1 \tag{19}$$

so that the return on deposits must at least compensate agents for inflation and their impatience.

Definition 1 *When banking is integrated, an unconstrained equilibrium is a list* $\{k_s, k^{\theta}, k, q, i, i_d\}$ *satisfying* (12)-(14)*,* (16)*,* (19) *and* $u'(q) = 1$*.*

Lemma 2 *Unconstrained equilibrium allocations are socially optimal at the Friedman rule, i.e.* $\gamma = \beta$*.*

At the Friedman rule, agents need no compensation for depositing cash and redeeming deposits. Since banks make zero profit, they charge no interest rate to entrepreneurs. This in turn allows entrepreneurs to invest the optimal amount of capital.

3.5.2 Constrained equilibrium

In a constrained equilibrium, agents consume such that $u'(q) > 1$ where q satisfies $M = d = pq$. Thus, (4) can be rewritten:

$$u'(q)(1+i) = \frac{\gamma}{\beta} \tag{20}$$

Definition 2 *A constrained equilibrium is a list* $\{k_s, k^\theta, k, q, i, i_d\}$ *satisfying* (12)-(14), (16), (20) *and* $q = c'(k_s)k_s(1+i)$.

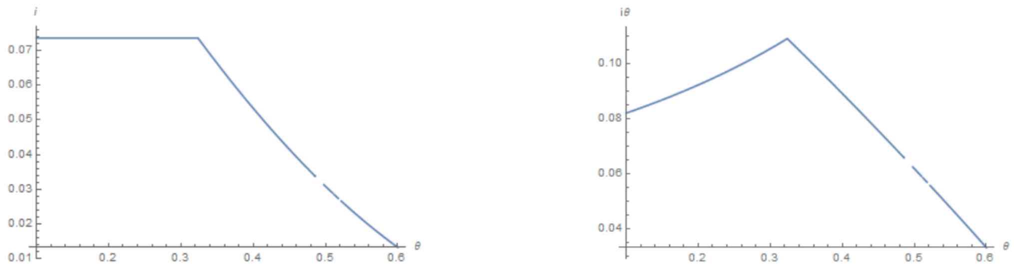

Figure 2: Interest rate on loans to safe and risky entrepreneurs

Figure 1 shows that equilibrium is unconstrained for low levels of θ and constrained for high levels of θ. When buyers are unconstrained, the interest rate on safe firms does not depend on θ. After a threshold level of θ, consumers become constrained since the technology is not productive enough, the interest rate on loans is then lowered by the liquidity premium on deposits. The interest rate on loans to risky entrepreneurs is increasing in θ when agents are unconstrained since entrepreneurs must pay a risk premium. If buyers are constrained then the liquidity premium lowers the interest rate.

Proposition 1 *With integrated banking, an unconstrained equilibrium exists if* $q^* \leq c'(k_s)\,k_s$ *and a constrained equilibrium exists if* $q^* > c'(k_s)\,k_s$*, where* $\{k_s, k\}$ *solves* $\beta f'(k) = c'(k_s)\,\gamma$ *and* $(1-\theta)\,\beta f'(2k_s - k) = c'(k_s)\,\gamma$*, and* $k_s < k$*.*

Consumers are unconstrained or constrained depending on the purchasing power of their holdings of bank deposits. In the integrated setting, bank deposits are equally remunerated in the risky and the safe country, thus consumers' purchasing power is the same in both countries. The real value of deposits is increasing in the amount of invested capital. If the real value of deposits is sufficiently high, consumers attain the socially efficient consumption quantity in the goods market.

Notice that for $\theta > 0$, the interest rate on risky loans is higher than the interest rate on safe loans. Hence from (12) and (13), $k > k^\theta$, implying $k_s < k$.

4 Fragmented banking

In this section we suppose that loan markets are fragmented. Banks can lend to their national firms only. Thus the interest rate on deposits is contingent on the state of nature. Let i_d be the nominal interest rate on deposits in the safe country and i_d^θ the one in the risky country. Suppliers still have the ability of selling to foreign firms. Settlement market programs and first order conditions remain the same. We will present the changes specifically for banks.

During the settlement market, banks face a different program since their investment choices will depend on the state of nature of the economy. A bank follows:

$$\max_d \frac{1}{2}\left[(1+i)\,\ell-(1+i_d)\,d\right]+\frac{1}{2}\left[(1-\theta)\left(1+i^\theta\right)\ell^\theta-\left(1+i_d^\theta\right)d+d-\ell^\theta\right]$$

subject to:

$$d \geq \ell$$
$$d \geq \ell^\theta$$

and depositors' participation constraint. Notice the program for the bank is slightly different from the one in the previous section. The term in the left bracket is the bank's profit when firms are safe. The term in the right bracket is the bank's profit when there is a positive aggregate default rate. In this case, a bank might not lend all its deposits since firms are less productive and keep $d-\ell$ idle fiat money balances. It would not be optimal for banks to acquire more deposits than what they would lend in the safe state, i.e. $d=\ell$.

Remember that banks cannot engage in cross-border lending and learn the state of the economy before the loans to entrepreneurs are made. Thus a bank in the risky state chooses the level of loan ℓ^θ following:

$$\max_{\ell^\theta}\,(1-\theta)\left(1+i^\theta\right)\ell^\theta-\left(1+i_d^\theta\right)d+\left(d-\ell^\theta\right)$$

Since we assume banks are perfectly competitive, the zero profit condition in the risky state is:

$$(1-\theta)\left(1+i^\theta\right)\ell^\theta-\left(1+i_d^\theta\right)d+\left(d-\ell^\theta\right)=0$$

where we have to consider two cases depending on the bank's resource constraint.

If the level of capital invested the risky country is lower than the one invested in the safe country, i.e. $k^\theta<k$, then a bank is indifferent between lending or keeping idle cash reserves when facing risky entrepreneurs. Thus it must be that the interest rate on loans and the one on cash reserves are equal such that:

$$i^\theta=(1-\theta)^{-1}-1 \tag{21}$$

and zero profit implies:

$$i_d^\theta=0 \tag{22}$$

since banks do not earn interest rate on loans since the continuum of firms is too risky.

If the level of invested capital is the same in both countries, i.e. $k = k^\theta$ then zero profit implies:

$$(1-\theta)\left(1+i^\theta\right) = 1 + i_d^\theta \tag{23}$$

such that the expected interest rate on loans is passed on to depositors. From (12)-(13) we deduce:

$$i = i^\theta \tag{24}$$

so the interest rates on loans are equal in both states in order to achieve the same level of invested capital.

4.1 Symmetric fragmented banking equilibrium

Now that banks do not face the same investment opportunities, the interest rates on deposits in both countries might be different. Since interest rates on deposits are contingent on the state of nature of the economy consumption in the goods market will differ from one country to the other. Thus, (4) rewrites:

$$u'(q)(1+i_d) + u'(q^\theta)(1+i_d^\theta) = 2\gamma/\beta \tag{25}$$

4.1.1 Unconstrained equilibrium

In an unconstrained equilibrium, agents consume such that $u'(q^\theta) = u'(q) = 1$, where q^θ is consumption in the country where firm default rate is positive and q where firms are perfectly safe. Since $u(q)$ is strictly monotonic, this means that $q^\theta = q$, so that agents consume the same quantities of goods in the goods market whatever the state. The first order condition on deposits (25) can be rewritten:

$$\frac{i_d^\theta + i_d}{2} = \frac{\gamma}{\beta} - 1 \tag{26}$$

where the left term is the expected interest rate on deposits, and the right side of the equation reflects the interest rate on money. We then have different subcases depending on lending when firms can default.

Definition 3 *A fragmented banking unconstrained equilibrium is a list* $\{k, k^\theta, k_s, q\}$ *satisfying* (12)-(14), $u'(q) = 1$ *and:*

i. $\{i, i_d, i^\theta, i_d^\theta\}$ *satisfy* (16), (23)-(24) *and* $i = \frac{1}{1-\theta/2}\frac{\gamma}{\beta} - 1$ *when the level of capital invested in both countries is the same, i.e.* $k = k^\theta$.

ii. $\{i, i_d, i^\theta, i_d^\theta\}$ *satisfy* (16), (21)-(22) *and* $i = 2(\gamma/\beta - 1)$ *when there is more investment in the safe country, i.e.* $k < k^\theta$.

When entrepreneurs have roughly the same productivity ($\theta < \bar{\theta}$), the technology is still good enough to yield a positive expected interest rate. Thus banks do not need to lend less to risky entrepreneurs than safe ones. The opposite effect happens when the level of risk is too high in the risky country, entrepreneurs are not productive enough such that banks prefer keeping idle cash balances.

Proposition 2 *There exists a critical value* $\bar{\theta} = \frac{\gamma-\beta}{\gamma-\beta/2}$*, such that:*

i. Denote $\bar{k}$ *the value of* k *solving* $f'(k) = \gamma c'(k)/\left[\beta(1-\theta/2)\right]$*, if* $\theta < \bar{\theta}$ *then an unconstrained equilibrium where* $k = k^{\theta} = \bar{k}$ *and* $q^{\star} < f'(\bar{k})\bar{k}(1-\theta)$ *exists.*

ii. Let $\tilde{k}$ *denote the value of* k *where* $\{k, k^{\theta}\}$ *solve* $(1-\theta)f'(k^{\theta}) = c'\left(\frac{k+k^{\theta}}{2}\right)$ *and* $f'(k) = c'\left(\frac{k+k^{\theta}}{2}\right)(2\gamma/\beta - 1)$*. If* $\theta > \bar{\theta}$ *then an unconstrained equilibrium where* $k > k^{\theta}$ *and* $q^{\star} < f'(\tilde{k})\tilde{k}\beta/(2\gamma-\beta)$ *exists.*

According to Proposition 1, when average firm default rate is high (θ close to 1) banks reach the point where keeping idle cash and investing is equivalent since firm production technology is too risky to yield a sufficient expected interest rate. Banks just compensate agents for inflation and impatience which they finance through a premium paid by entrepreneurs in the safe state which lowers the level of capital invested by entrepreneurs since loans are marginally more expensive. Notice that at the Friedman rule, $\bar{\theta} = 0$, such that an equilibrium where $k = k^{\theta}$ does not exist since holding cash is not costly for buyers.

Proposition 3 *In an unconstrained equilibrium with fragmented banking, for any positive loan interest rate (i.e.,* $\gamma > \beta$*), capital allocation in each country satisfies* $K > k$ *and* $K^{\theta} < k^{\theta}$*, while aggregate capital investment satisfies* $k_s^p > k_s$*.*

Proposition 3 states that the overall level of invested capital is lower than the social planner's. Capital investment satisfies $K > k$ in the safe country and $K^{\theta} < k^{\theta}$ in the risky country, implying that there is overinvestment in the risky country compared to social optimum.

4.1.2 Constrained equilibrium

A constrained equilibrium is such that $u'(q^{\theta})$, $u'(q) > 1$. In a constrained equilibrium with fragmented banking, the first order conditions on deposits rewrites:

$$u'(q^{\theta})(1 + i_d^{\theta}) + u'(q)(1 + i_d) = 2\frac{\gamma}{\beta} \tag{27}$$

since we assume buyers deposit all their money balances it must be that $d = M$. Furthermore, since buyers are constrained, then consumption satisfies $p^{\theta}q^{\theta} = pq = d = M$.

Definition 4 *A fragmented banking constrained equilibrium is a list* $\{k, k^{\theta}, k_s, q\}$ *satisfying* (12)-(14)*,* $q = f'(k)k$*,* $q^{\theta} = c'(k_s)k(1 + i_d^{\theta})$ *and:*

i. $\{i, i_d, i^{\theta}, i_d^{\theta}\}$ *satisfy* (16)*,* (23)-(24) *and* $i = \frac{2\gamma}{\beta\left[(1-\theta)u'(q^{\theta})+u'(q)\right]} - 1$ *when the level of capital invested in both countries is the same, i.e.* $k = k^{\theta}$*.*

ii. $\{i, i_d, i^{\theta}, i_d^{\theta}\}$ *satisfy* (16)*,* (21)-(22) *and* $i = \frac{2\gamma-\beta u'(q^{\theta})}{\beta u'(q)} - 1$ *when there is more investment in the safe country, i.e.* $k > k^{\theta}$*.*

Agents are unconstrained in the good state i.e. $u'(q) = 1$ and constrained in the bad state i.e. $u'(q^\theta) > 1$ and so $q > q^\theta$. From depositors' optimal condition we get

$$\frac{1}{2}[(1+i_d) + u'(q^\theta)(1+i_d^\theta)] = \frac{\gamma}{\beta} \tag{28}$$

where $q^\theta = d/p^\theta = M\phi(1+i_d^\theta)$.

Definition 5 *A fragmented banking constrained/unconstrained equilibrium is a list* $\{k, k^\theta, k_s, q\}$ *satisfying* (12)-(14), $q = u'^{-1}[1]$, $q^\theta = c'(k_s)k(1+i_d^\theta)$ *and:*

i. $\{i, i_d, i^\theta, i_d^\theta\}$ *satisfy* (16), (23)-(24) *and* $i = \frac{2\gamma}{\beta[1+(1-\theta)u'(q^\theta)]-1}$ *when the level of capital invested in both countries is the same, i.e.* $k = k^\theta$.

ii. $\{i, i_d, i^\theta, i_d^\theta\}$ *satisfy* (16), (21)-(22) *and* $i = 2\gamma/\beta - u'(q^\theta) - 1$ *when there is more investment in the safe country, i.e.* $k > k^\theta$.

In order to further characterize the constrained equilibrium with fragmented banking markets, we first present the deposit insurance scheme.

5 Deposit insurance

To motivate a role for deposit insurance we assume that the state of nature is revealed at the beginning of a new period, such that bank deposit contracts are made before. This motivates a role for deposit insurance that we model as a contract where the bank in the country with the lowest θ deposits money into a fund to compensate the other bank from losses.[8] Countries are ex-ante symmetrical. Banks write a deposit insurance contract before the realization of the shock which forces the bank in the "good" state to cover the losses of the one in the bad state. The motivation for the deposit insurance contract is that depositors are better off having the same interest rate i_d in both states such that consumption will be the same independently of the shock i.e. $q_h^\theta = q$ as the interest rate affects the price of consumption goods in the goods market. The contributions have to be incentive compatible, i.e. non negative expected profits, such that a bank sets its contribution τ following:

$$\frac{1}{2}\left[(1-\theta)\left(1+i^\theta\right)\ell^\theta - (1+i_d)\,d + d - \ell^\theta + (1+i_d)\,d - (1-\theta)\,\ell^\theta\left(1+i^\theta\right) - \left(d-\ell^\theta\right)\right] + \frac{1}{2}\left[(1+i)\,\ell - (1+i_d)\,d - \tau d\right] = 0$$

where the first term between brackets is the profit when the bank is in the good state, such that it makes profit from the spread between the interest rate on loans and deposits, which has to finance the deposit insurance for the other bank. $(1+i_d)\,d -$

[8] Some could argue that deposit insurance are funds deposited before investing (costly idle liquidity). For example, banks in the model would not have d deposits to lend but $(1-\tau)d$ deposits, where τ would be the money deposited into the fund. This would not change the main insights of the model and just complexify calculations.

$(1-\theta)\,\ell^{\theta}\left(1+i_{\ell}^{\theta}\right)-\left(d-\ell^{\theta}\right)$ is the payment received from the fund when banks face risky entrepreneurs (with probability 1/2).

Rearranging, we obtain

$$-\tau d+\left[(1+i)\,\ell-(1+i_d)\,d\right]=0$$

So with deposit insurance, the banks potential losses are entirely covered by the fund, and so profit in bad state is 0. Since $\ell^{\theta}=d$, the equation above therefore rewrites:

$$\tau=i-i_d \tag{29}$$

Countries are ex-ante the same, so $\tau_h=\tau_f=\tau$. In our current set-up there is no uncertainty about one country defaulting. It should be optimal to have a fund that just covers the amount of the losses such that we get:

$$\tau d=(1+i_d)d-(1-\theta)(1+i^{\theta})l^{\theta}-(d-\ell^{\theta}) \tag{30}$$

Since now depositors get the same interest rate on deposits, the optimal condition on deposits becomes:

$$\frac{\gamma}{\beta}=u'(q)\,(1+i_d) \tag{31}$$

When there is no credit rationing, i.e. $l^{\theta}<d$ then banks being indifferent between keeping deposits and lending implies $(1-\theta)(1+i^{\theta})=1$, which combined with (30) yields:

$$\tau=i_d$$

This is the situation where investing in risky firms yields the same return as keeping cash balances intra period. Banks facing risky firms do not earn a positive spread on loans, thus safe firms finance the interest rate on deposits through the loan rates, thus using (29) the interest rate on safe firm loans follows:

$$i=2i_d \tag{32}$$

When banks in both states grant the same nominal amount of loans, i.e. $\ell^{\theta}=\ell=d$, firm first order conditions, it implies $i=i^{\theta}$. Hence (30) gives:

$$\tau=1+i_d-(1-\theta)(1+i^{\theta})$$

which combined with (29) yields:

$$i=\frac{2i_d+\theta}{2-\theta} \tag{33}$$

The spread finances total unpaid loans. In this equilibrium $i^{\theta}=i>(1-\theta)^{-1}-1$: the bank strictly prefers lending all its deposits out even in the bad state.[9]

[9] Consider the extreme case $\theta=1$. In this case $\tau=\frac{1}{2}i_d$ which combined with $\tau=\frac{1}{2}(i-i_d)$ gives $\frac{1}{2}i_d=\frac{1}{2}(i-i_d)\implies i_d=i-i_d\implies i_d=\frac{i}{2}$. If $\theta=0$ then $i_d=i=i^{\theta}$.

5.1 Unconstrained equilibrium

In an unconstrained equilibrium with deposit insurance, agents obtain the same interest rate on deposits whether in good or bad state and consume q such that $u'(q) = 1$. We can deduce from (31):

$$i_d = \gamma/\beta \tag{34}$$

Definition 6 *An unconstrained fragmented banking equilibrium with deposit insurance is a list $\{k, k^\theta, k_s, q\}$ satisfying (12)-(14), $q = u'^{-1}[1]$ and:*

i. $\{i, i_d, i^\theta\}$ satisfy (24), (33) and $i = \frac{\gamma}{(1-\theta/2)\beta} - 1$ when the level of capital invested in both countries is the same, i.e. $k = k^\theta$.

ii. $\{i, i_d, i^\theta\}$ satisfy (21), (32)and $i = 2\left(\frac{\gamma}{\beta} - 1\right)$ when there is more investment in the safe country, i.e. $k > k^\theta$.

Proposition 4 *There exists a critical value $\bar{\theta} = \frac{\gamma-\beta}{\gamma-\beta/2}$, such that:*

i. Denote $\bar{k}$ the value of k solving $f'(k) = \gamma c'(k)/\left[\beta(1-\theta/2)\right]$, if $\theta < \bar{\theta}$ then an unconstrained equilibrium where $k = k^\theta = \bar{k}$ and $q^\star < f'(\bar{k})\bar{k}(1-\theta/2)$ exists.

ii. Let $\tilde{k}$ denote the value of k where $\{k, k^\theta\}$ solve $(1-\theta)f'(k^\theta) = c'\left(\frac{k+k^\theta}{2}\right)$ and $f'(k) = c'\left(\frac{k+k^\theta}{2}\right)(2\gamma/\beta - 1)$. If $\theta > \bar{\theta}$ then an unconstrained equilibrium where $k > k^\theta$ and $q^\star < f'(\tilde{k})\tilde{k}\gamma/(2\gamma-\beta)$ exists.

Proposition 5 *With a common deposit insurance allows, the unconstrained equilibrium exists for a larger set of values of the productivity shock.*

Deposit insurance in our model does not modify lending rates. However, the loan rate in the good state finances all losses in the bad state to ensure an equal deposit rate in both states. As such we have equal loan rates with and without deposit insurance and $i_d > i_d^\tau > i_d^\theta$ (where i_d^θ and i_d are respectively the deposit rates without and with deposit insurance). Thus, the levels of capital are the same in both models (given by (12)-(13)) while expected utility of consumption is improved because of the smoothing allowed by the constant deposit rate. We see in the case where $\theta > \bar{\theta}$ that even though capital levels are the same (because the level of interest rates on loans are the same), the threshold for consumption to be unconstrained is different, as long as $\gamma > \beta$, consumption with deposit insurance will be unconstrained for more values of productivity shocks.

5.2 Constrained equilibrium

In the constrained equilibrium with deposit insurance, consumption is equal in both states but agents being constrained, marginal utility follows $u'(q) > 1$.

Definition 7 *An constrained fragmented banking equilibrium with deposit insurance is a list* $\{k, k^\theta, k_s, q\}$ *satisfying* (12)-(14), $q = f'(k)k(1+i_d)/(1+i)$ *and:*

i. $\{i, i_d, i^\theta\}$ *satisfy* (24), (33) *and* $i = \frac{\gamma}{(1-\theta/2)\beta u'(q)} - 1$ *when the level of capital invested in both countries is the same, i.e.* $k = k^\theta$*.*

ii. $\{i, i_d, i^\theta\}$ *satisfy* (21), (32)*and* $i = 2\left(\frac{\gamma}{\beta u'(q)} - 1\right)$ *when there is more investment in the safe country, i.e.* $k > k^\theta$*.*

Proposition 6 *There exists a critical value* $\tilde{\theta} = \frac{\gamma - \beta u'(q)}{\gamma - \beta u'(q)/2}$*, such that:*

i. Let $\bar{k}$ *denote the value of* k *where* $\{k, q\}$ *solve* $q = f'(k)k(1-\theta/2)$ *and* $\beta u'(q) = c'(k)\gamma/(f'(k)(1-\theta/2))$*, if* $\theta < \tilde{\theta}$ *and* $f'(\bar{k})\bar{k}(1-\theta/2) < q^\star$ *then a constrained equilibrium with deposit insurance where* $k = k^\theta$ *exists.*

ii. Let $\tilde{k}$ *and* $\tilde{q}$ *denote respectively the values of* k *and* q *where* $\{k, k^\theta, q\}$ *solve* $(1-\theta)f'(k^\theta) = c'\left(\frac{k+k^\theta}{2}\right)$, $f'(k) = c'\left(\frac{k+k^\theta}{2}\right)\left(\frac{2\gamma - \beta u'(q)}{\beta u'(q)}\right)$ *and* $q = \frac{f'(k)k\gamma}{2\gamma - \beta u'(q)}$*. If* $\theta > \tilde{\theta}$ *and* $\tilde{q} < q^\star$ *then an unconstrained equilibrium with deposit insurance where* $k > k^\theta$ *exists.*

Proposition 7 *With fragmented banking, a common deposit insurance eliminates consumption risk sharing* ($q = q^\theta$)*, in both the unconstrained and constrained equilibria. Capital allocation is suboptimal (in the unconstrained equilibrium, the deposit insurance leaves the capital allocation unaffected).*

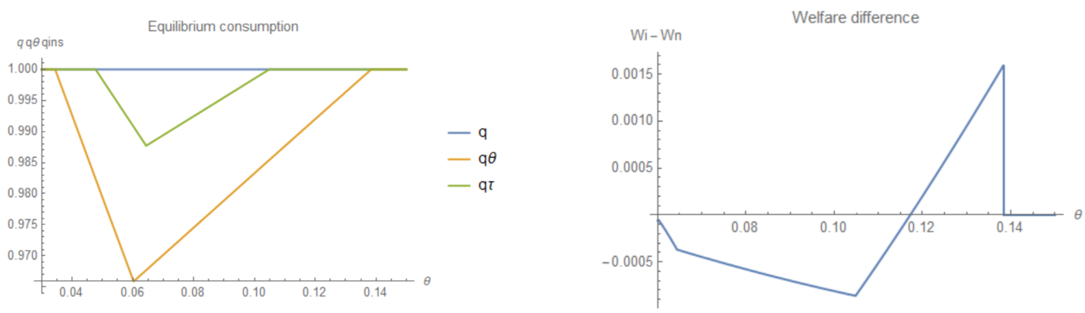

Figure 3: Consumption and welfare with and without deposit insurance

As Proposition 7 states, a common deposit insurance allows consumption smoothing by making bank deposits riskless. However, as it can be deduced from the bank's problem, with fragmented banking a common deposit insurance does not improve resource allocation on the lending side, implying lower social welfare than in the integrated banking setting. Moreover, as Figure 3 shows for a numerical example with fragmented banking markets, the negative effect of a distorted capital allocation on

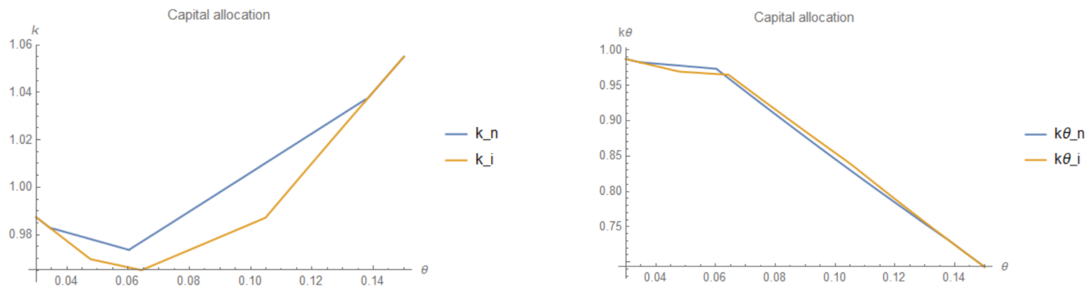

Figure 4: Capital allocation with and without deposit insurance

welfare caused by a common deposit insurance may outweigh the positive effect of consumption smoothing. Since the deposit insurance causes the level of invested capital by riskless firms to shrink, risky firms benefit from a lower price of capital and are able to increase their capital investment. This inefficient redistribution of capital reduces aggregate output. As a result, even if consumption risk is completely eliminated, welfare may be higher without deposit insurance. For intermediate values of θ the capital distortion is attenuated because banks are less willing to grant credit to risky firms. For these levels of θ, welfare is thus higher with a common deposit insurance.

6 Central Bank Digital Currency

In this section we introduce Central Bank Digital Currency as a new means of payment replacing cash in the previous model and that can be a perfect substitute to bank deposits in good market transactions. We assume that deposit balances can be redeemed in terms of CBDC balances on the spot at the beginning of the goods market. That is, if buyers hold low return deposits, they can change them into central bank money in order to purchase more goods. Let E be the quantity of CBDC, growing at a rate γ. Thus, $W(e, d)$ is the value of being a buyer entering the setllement market with e nominal CBDC balances and d deposit balances. In the country hit by the productivity shock, the program is the following:

$$W^\theta(m, d) = \max_{x, e_{+1}, d_{+1}} x + \beta \left[\frac{1}{2} V(m_{+1}, e_{+1}) + \frac{1}{2} V^\theta(e_{+1}, d_{+1}) \right]$$

subject to

$$x + \phi(e_{+1} + d_{+1}) = h + \phi(1 + i_d^\theta)d + \phi e(1 + i^e)$$

which yields the following FOC on CBDC holdings:

$$\beta V_e(e, d) \leq \phi_{-1} \tag{35}$$

In the goods market, the value for buyers follows

$$V^\theta(e, d) = \max_{q^\theta, q_e^\theta, q_d^\theta} u(q^\theta) + W^\theta(e - p^e q_e^\theta, d - p^\theta q_d^\theta)$$

subject to

$$e + d \geq p^e q_e^\theta$$

$$d + e - p^e q_e^\theta \geq p^\theta q_d^\theta$$
$$q_d^\theta + q_e^\theta = q^\theta$$

where q_e^θ is the quantity of goods purchased with CBDC. The intuitition is very simple. If the return on CBDC is higher than the one on deposits, i.e. $i^e > i_d^\theta$ then buyers will exchange their deposits for CBDC in order to increase their purchasing power. If returns are equal then they are indifferent between exchanging deposits for CBDC.

Agents having an outside option to deposits in the goods market, banks have to make sure that buyer's do not have an incentive to ask for their CBDC balances back. Since loans are made before the goods market opens, banks would be unable to meet buyer liquidity demand in case of withdrawal. As such the interest rate on deposits in the bad state has to be at least equal to the one on CBDC, i.e. $i_d^\theta \geq i^e$. The same result is found by using the no arbitrage condition between keeping CBDC balances and earning interest rate on loans. Banks thus invest until the expected interest rate on loans is the same as the one on CBDC, i.e. $i^e = (1-\theta)(1+i^\theta)$. Thus, the program for banks when CBDC is introduced can be rewritten:

$$\max_d \frac{1}{2}\left[(1+i)\,\ell - (1+i_d)\,d\right] + \frac{1}{2}\left[(1-\theta)\left(1+i^\theta\right)\ell^\theta - \left(1+i_d^\theta\right)d + \left(d-\ell^\theta\right)(1+i^e)\right]$$

subject to:

$$d \geq \ell$$
$$d \geq \ell^\theta.$$

In the risky state, the bank chooses the loan level ℓ^θ such that it maximises its profits following:

$$\max_{\ell^\theta} (1-\theta)(1+i^\theta)\ell^\theta - (1+i_d^\theta)d + (d-\ell^\theta)(1+i^e)$$

subject to $d \geq \ell^\theta$. The first order condition yields $(1-\theta)(1+i^\theta) = 1 + i^e + \lambda_d$ where λ_d is the Lagrangian multiplier on the loan constraint. If the constraint is binding, i.e. $\lambda_d > 0$ then the expected interest rate on loans is higher than the one on CBDC, i.e. $(1-\theta)(1+i^\theta) > 1+i^e$ and the bank lends all its deposits to entepreneurs. In the event where the entrepreneurs are not productive enough, then the constraint is not binding and banks keep idle CBDC balances in their portfolio such that $(1-\theta)(1+i^\theta) = 1+i^e$ and $\ell^\theta < d$.

If banks face safe entrepreneurs, they lend following:

$$\max_\ell (1+i)\ell - (1+i_d)d$$

subject to $d = \ell$ (since as we saw in previous sections, banks will never acquire more deposits than what they can lend in the safe state). So from zero profit condition we still obtain $i_d = i$.

Now let us turn to the FOC on deposit and CBDC balances. Using the envelope conditions, we have :

$$(1+i^e)u'(q^e) \leq \gamma/\beta \tag{36}$$

$$(1 + i_d)u'(q) + (1 + i_d^\theta)u'(q^\theta) = 2\gamma/\beta \tag{37}$$

so we see that using CBDC to purchase in the goods market has the advantage of smoothing consumption perfectly. However, since banks lend just the level of capital such that their is no arbitrage between keeping CBDC and lending in the bad state, zero profit condition for banks in state θ implies $1 + i_d^\theta = 1 + i^e$. From seller first order conditions this implies that the price of consumption is the same using CBDC or deposits such that $q^e = q^\theta$, since agents have the same purchasing power using deposits or CBDC.

Let us look at what happens in an unconstrained equilibrium, i.e. $u'(q^\theta), u'(q) = 1$. The loans rates follow $1 + i^\theta = (1 + i^e)/(1 - \theta)$ and $1 + i_d = 2\gamma/\beta - (1 + i^e)$, such that we get the following equilibrium conditions:

$$\begin{aligned}
(1-\theta)f'(k^\theta) &= (1 + i^e)c'(k_s) \\
f'(k) &= [2\gamma/\beta - (1 + i^e)]\, c'(k_s) \\
k + k^\theta &= 2k_s \\
i &= i_d = 2(\gamma/\beta - 1) - i^e \\
i^\theta &= (1 + i^e)(1 - \theta) - 1
\end{aligned}$$

With fragmented markets, in the unconstrained equilibrium k^θ is smaller with a interest-bearing CBDC than with no CBDC. This entails a decrease in k_s and an increase in k. Since demand for capital decreases in the risky country, the price of capital decreases thereby favoring higher capital investment in the safe country. In addition, the deposit and loan rates in the safe country, i_d and i are smaller with CBDC, which also favors investment in the safe country. The introduction of CBDC then mitigates capital misallocation.

7 Concluding remarks

In this paper we have presented a two-country model of monetary exchange and banking where inversely correlated country-specific shocks provide a motive for risk sharing within a currency union. We find that introducing a common deposit insurance when retail credit markets are fragmented has a positive effect on welfare by smoothing depositors' consumption. We find similar results when a central-bank digital currency is introduced. However, a central-bank digital currency may be more more effective than a common deposit insurance in promoting efficient capital allocation within the currency union.

Our paper abstracts from many institutional details that are relevant for the success of these policies. Deposit insurance implies full commitment from banks. In the absence of political will for introducing deposit insurance, introducing a central-bank digital currency would be beneficial for risk sharing. Our model is open to many extensions such as extending the analysis for asymmetric productivity shocks, considering a non-competitive banking sector, introducing moral hazard considerations or allowing for an interbank market.

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Appendix

Proof of Proposition 1. For this equilibrium to exist, q must be equal to q^* with $u'(q^*) = 1$. Hence we need $pq^* \leq d = M \implies \phi pq^* \leq \phi M \implies q^* \leq \phi M$. Banks lend all cash out to entrepreneurs. Market clearing condition in the market for loans yields $\frac{\ell+\ell^\theta}{2} = M$. In addition $\ell = p_k k = \frac{c'(k_s)}{\phi}k \implies \phi\ell = c'(k_s)k$ and $\ell^\theta = p_k k^\theta = \frac{c'(k_s)}{\phi}k^\theta \implies \phi\ell^\theta = c'(k_s)k^\theta$.

Combining with the market clearing condition in the loans market, it yields $c'(k_s)\frac{k+k^\theta}{2} = \phi M$ that we can further write as $c'(k_s)k_s = \phi M$. Hence for buyers to be unconstrained, we need: $q^* \leq c'(k_s)k_s$. We also have $f'(k) = c'(k_s)\gamma/\beta$ and $(1-\theta)f'(k^\theta) = c'(k_s)\gamma/\beta \implies \beta f'(k) = c'(k_s)\gamma$ and $(1-\theta)\beta f'(2k_s - k) = c'(k_s)\gamma$ with $k^\theta < k_s < k$. This last inequality implies that the value of k that solves $\beta f'(k) = c'(k_s)\gamma$ for $\gamma = 1$ verifies $k > \bar{k}$. If further θ is sufficiently small, k^θ is big enough so that k^s is sufficiently close to $\bar{k}$. Then if γ is sufficiently small, k_s is sufficiently close to $\bar{k}$ and the unconstrained equilibrium exists. Alternatively, if for a given γ the scalar a is sufficiently small the unconstrained equilibrium exists.

Notice that if $c'(k) = 1$ the equilibrium condition for $\gamma = 1$ simplifies to $q^* \leq \bar{k}_s$ where $\bar{k}_s = \frac{\bar{k}+\bar{k}^\theta}{2}$ and $\bar{k}^\theta$ solves $(1-\theta)\beta f'(\bar{k}^\theta) = 1$. Since $\bar{k}^\theta < \bar{k}$, we have that $\bar{k}_s < \bar{k}$. Therefore, with a linear cost c, the region of existence of the unconstrained equilibrium shrinks for $\theta > 0$ compared to $\theta = 0$.

Proof of Proposition 2. $k = k^\theta$ (banks lend all deposit balances in both countries) if the expected interest rate on risky loans is higher than the return on money, i.e. $(1-\theta)(1+i^\theta) > 1$. From firm FOCs (12)-(13) it follows that the interest rate on loans must be equal, i.e. $i^\theta = i$. Zero profit condition yields $(1-\theta)(1+i^\theta) = 1 + i_d^\theta$ and $i_d = i$. It follows from first order condition on deposits (25) that $1 + i_d = \frac{1}{1-\theta/2}\frac{\gamma}{\beta}$. Since the expected interest on risky loans has to be greater than the return on money, this equilibrium exists if $\frac{1-\theta}{1-\theta/2}\frac{\gamma}{\beta} > 1 \implies \theta < \frac{\gamma-\beta}{\gamma-\beta/2} = \bar{\theta}$. If buyers are unconstrained, then $q < \phi M(1+i_d^\theta)$. From (14),$k = k^\theta \implies k = k^\theta = k_s$. Since $p_k k = M \implies \phi = c'(k)k/M$, $q < \phi M(1+i_d^\theta) \implies q < c'(k)k(1+i_d^\theta)$. Since $(1-\theta)(1+i_d^\theta) = 1+i$ and from (13), $c'(k) = f'(k)/(1+i)$, our condition on q rewrites $q^\star < f'(k)k(1-\theta) \implies u'^{-1}[1] < f'(k)k(1-\theta)$, where k solves $f'(k) = \frac{1}{1-\theta/2}\frac{\gamma}{\beta}c'(k)$. If $k > k^\theta$ (banks lend less in the risky country), from (12)-(13), $k > k^\theta \implies i^\theta > i \implies \frac{1}{1-\theta} > 2\frac{\gamma}{\beta} - 1 \implies \theta > \bar{\theta}$. We need to find the condition for which buyers are unconstrained, i.e. $q^\star = u'^{-1}[1]$ and $q^\star < M/p^\theta \implies q^\star < \phi M(1+i_d^\theta)$. Since $c'(k_s) = \phi p_k \implies p_k = c'(k_s)/\phi$ and $\ell = M = p_k k \implies p_k = M/k$, we obtain $\phi = c'(k_s)k/M$. So $q^\star$ must satisfy $q^\star < c'(k_s)k$. From (13) we have $f'(k) = (1+i)c'(k_s)$, from which we deduce $q^\star < f'(k)k/(1+i) \implies q^\star < f'(k)k\beta/(2\gamma-\beta)$ where $\{k, k^\theta, k_s\}$ satisfy (12)-(14).

Proof of Proposition 3.

i. if $K < k$ and $K^\theta < k^\theta$ then from market clearing condition $k_s^p < k_s$ while $f'(K) > f'(k) \implies c'(k_s^p) > c'(k_s) \implies k_s^p > k_s$ which is inconsistent with the other inequality. Capital allocations in this equilibrium cannot be greater than social planner's for both types of firms.

ii. if $K > k$ and $K^\theta < k^\theta$ then $f'(K^\theta) > f'(k^\theta) \implies c'(k_s^p) > c'(k_s) \implies k_s^p > k_s$ and $f'(K) < f'(k) \implies c'(k_s^p) < \left(\frac{2\gamma}{\beta} - 1\right) c'(k_s) \implies k_s^p > k_s$ if $1 < \frac{c'(k_s^p)}{c'(k_s)} < \left(\frac{2\gamma}{\beta} - 1\right)$.

iii. if $K < k$ and $K^\theta > k^\theta$ then $f'(K) > f'(k) \implies c'(k_s^p) > c'(k_s)\left(\frac{2\gamma}{\beta} - 1\right)$ and thus $k_s^p > k_s$ since we assume $\gamma \geq \beta$. The other condition yields $f'(K^\theta) < f'(k^\theta) \implies c'(k_s^p) < c'(k_s) \implies k_s^p < k_s$. So the initial assumptions cannot be satisfied in equilibrium.

iv. if $K > k$ and $K^\theta > k^\theta$ then $f'(K^\theta) < f'(k^\theta) \implies c'(k_s^p) < c'(k_s) \implies k_s^p < k_s$. On the other hand, using (14), the initial assumptions imply $k_s^p > k_s$ which is inconsistent.

Proof of Proposition 4.

i. if $k = k^\theta$, from (12)-(13) we obtain $i = i^\theta$. From $i = (2i_d + \theta)/(2 - \theta)$ and using (31), we obtain $i = 2\gamma/[(2-\theta)\beta] - 1 = \gamma/[(1-\theta/2)\beta] - 1$. We then have to check that agents are unconstrained, i.e. $q^\star < c'(k)k(1+i_d) \implies q^\star < f'(k)k(1+i_d)/(1+i)$. Since $1 + i_d = \gamma/\beta$ and $1 + i = \gamma/(\beta(1-\theta/2))$, $(1+i_d)/(1+i) = 1 - \theta/2$, thus agents are indeed unconstrained as long as $q^\star < f'(k)k(1-\theta/2)$, where k solves $f'(k) = \gamma c'(k)/(\beta(1-\theta/2))$.

ii. if $k > k^\theta$, from (32), we obtain $i = 2(\gamma/\beta - 1)$. While the indifference condition yields: $i^\theta = (1-\theta)^{-1} - 1$. Since $\ell^\theta < d \Rightarrow \ell^\theta < \ell \Rightarrow k > k^\theta$, using firm first order conditions we must have $i < i^\theta \Leftrightarrow \gamma/\beta < \frac{1-\theta/2}{1-\theta} \Leftrightarrow \theta > \bar{\theta}$. We also have to check that depositors are constrained, i.e. $q^\star \leq c'(k_s)k(1+i_d)$, which using (13) rewrites $q^\star < f'(k)k(1+i_d)/(1+i)$. Since $i_d = i/2 \implies i_d = \gamma/\beta - 1$, the inequality rewrites $q^\star \leq f'(k)k(\gamma/\beta)/(2\gamma/\beta - 1) \implies q^\star \leq f'(k)k\gamma/(2\gamma - \beta)$, where k is obtained through $\{k, k^\theta\}$ solving $(1-\theta)f'(k^\theta) = c'(\frac{k+k^\theta}{2})$ and $f'(k) = (2\gamma/\beta - 1)c'(\frac{k+k^\theta}{2})$.

Proof of Proposition 6.

i. if $k = k^\theta$ then $(2i_d + \theta)/(2 - \theta)$. Since buyers are constrained, $u'(q) > 1$, thus from (31) we obtain $1 + i = \frac{\gamma}{\beta u'(q)(1-\theta/2)}$. Thus using (13) and (31) and since $q = c'(k)k(1+i_d)$, we have $\{q, k\}$ satisfying $q = f'(k)k(1-\theta/2)$ and $f'(k) = \frac{c'(k)\gamma}{\beta u'(q)(1-\theta/2)}$. Thus agents are constrained if $f'(k)k(1-\theta/2) < q^\star$. We must also check that expected interest rate is incentive compatible for banks, i.e. $(1-\theta)(1+i) \implies \theta < \frac{\gamma - \beta u'(q)}{\gamma - \beta u'(q)/2}$.

ii. if $k > k^\theta$ then $i = 2i_d$. Since $q = c'(k_s)k(1+i_d)$, we can rewrite using (13) $q = f'(k)k(1+i_d)/(1+i)$. Using (31) we obtain $q = f'(k)k\gamma/(2\gamma - \beta u'(q)) < q^{star}$, so that buyers are constrained. We also have to check that the interest rates on loans are compatible with $k > k^\theta$, which using (12)-(13) is equivalent to $i < i^\theta \implies \theta > \frac{\gamma - \beta u'(q)}{\gamma - \beta u'(q)/2}$.